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# Trinity Matrices, Properties and Construction

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**Abstract:** In order to study the Kevin Helmholtz Instability waves and upstream propagated acoustic waves for supersonic multiple jets, we have to solve Bessel Equations in the domain of multiple-cylinders and derive dispersion relations to interpret jet flow. The Trinity Matrices play a very significant role in the analysis. Trinity Matrices are special kind of circulant, symmetric, and non-singular matrixes. They have all the properties of circulant matrices, but with more unique properties. Trinity Matrices can also be used for studying quadratics in n-dimensional spaces; matrix functions, etc. For any integer larger than or equal to 3, $n \geq 3$, the n by n Trinity Matrices always exist. The numbers of Trinity Matrices increase if the dimension of the matrix becomes larger. After giving definition of Trinity Matrices, the paper provides some properties of these matrices. Author also provides many examples in this paper. From these properties and examples, author sincerely hopes readers can make some suggestions, more information, and criticisms.

**Key words:** circulant matrices, trinity matrices.

## 1. Definition

A matrix $A = (a_{ij})_{n,n}$ is called as trinity matrix if it satisfied following properties:

$$a_{i+1,j+1} = a_{i,j}, \quad A = A^{-1} = A^T, \quad \prod_{i,j=1}^{n} a_{i,j} \neq 0$$

We noticed that

(1) The Trinity matrix $A = (a_{ij})_{n,n}$ with $a_{i+1,j+1} = a_{i,j}$ is a *Toeplitz* matrix.

$$A = Circulant(a_1, a_2, a_3, \ldots, a_n) = \begin{pmatrix} a_1 & a_2 & a_3 & . & a_n \\ a_n & a_1 & a_2 & . & a_{n-1} \\ a_{n-1} & a_n & a_1 & . & a_{n-2} \\ . & . & . & . & . \\ a_2 & a_3 & a_4 & . & a_1 \end{pmatrix}$$

(2) It satisfies $A = A^{-1} = A^T$.

(3) The condition of $\prod_{i=1}^{n} a_i \neq 0$ excludes identity matrices $I_n$ from Trinity Matrices.

## 2. Properties of Trinity Matrices

The trinity matrices are special *Toeplitz* matrices; hence trinity matrices have all properties of *Toeplitz* matrices have. We can find following special properties of Trinity Matrices

(1) $a_1, a_2, a_3, \ldots, a_n$ satisfies

$$a_1^2 + a_2^2 + a_3^2 + \ldots + a_n^2 = 1$$

$$a_1 a_{k+1} + \ldots + a_{n-k} a_n + a_{n-k+1} a_1 + \ldots a_n a_k = 0$$

Where $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$, $\lfloor x \rfloor$ is the Greatest Integer Function of which is less than or equal to $x$.

Furthermore, since $A = A^T$, the entries of matrix $A$ have the following relations:

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- If n is even, $n = 2k, k = 2, 3, \ldots$,

then $a_i = a_{2k+2-i}, i = k+2, k+3, \ldots 2k$.

- If n is odd, $n = 2k+1, k = 2, 3, \ldots$,

then $a_n = a_2, a_{n-1} = a_3, a_{n-2} = a_4, \ldots, a_{k+2} = a_{k+1}$.

(2) $a_1, a_2, a_3, \ldots, a_n$ Satisfies the following formula,

$$\sqrt{a_1^2 + a_2^2 + a_3^2 + \ldots + a_n^2} = a_1 + a_2 + a_3 + \ldots + a_n = 1$$

It is called the Freshman's Dream formula, the terminology due to V. O. McBrien [1].

(3) The following norms of trinity matrices are

$$\| A \|_1 = \| A \|_\infty = \sum_{i=1}^{n} | a_i | > \sum_{i=1}^{n} a_i = 1 \ ; \ \| A \|_2 = 1,$$

$$\| A \|_F = n\sqrt{\sum_{i=1}^{n} a_i} = n .$$

(4) $\lambda = 1$, $\lambda = -1$ are the only eigenvalues of $A$, hence the spectrum radius of all Trinity Matrices are $\rho(A) = 1$.

In fact, assume that there exists nonzero vector $\vec{x} \in R^n$ such that $A\vec{x} = \lambda\vec{x}$ then we have

$$A\vec{x} = \lambda\vec{x} = \lambda A^2 \vec{x} = \lambda A(A\vec{x}) = \lambda^2 (A\vec{x})$$
$$\lambda^2 = 1$$

(5) Trinity matrices are orthogonally diagonalizable matrices as $A = PDP^T$, where $P$ is orthogonal matrix, $P^{-1} = P^T$, $D$ is diagonal matrix and $d_{i,i} = \lambda_i = \pm 1$.

(6) The trace of trinity matrix is

$$Trace(A) = \sum_{i=1}^{n} a_{i,i} = na_1 .$$

(7) The characteristic polynomial of any trinity matrix is in the form $p_n(\lambda) = (\lambda - 1)^k (\lambda + 1)^{n-k}, \quad k = 1, 2, \ldots, n$.

For each $k = 1, 2, \ldots, n$, we can find trace of A from $p_n(\lambda)$ by using the coefficient of $\lambda^{n-1}$ term in the polynomial $p_n(\lambda)$. We can find $a_1 = \pm \frac{1}{n} Trace(A)$.

(8) For any integer $n \geq 3$, the numbers of $n \times n$ trinity matrices are limited; the number of trinity matrices increases when $n$ increases.

(9) There is no 2 by 2 Trinity Matrix.

(10) For any $n \geq 3$, trinity matrices exist. For example,

$$A_{n,n} = \pm \frac{1}{n} Circulant(2-n, \quad 2, \quad 2, \quad 2, \quad \ldots\ldots, \quad 2).$$

## 3. Preparation for Construction of Trinity Matrices

Find $a_1$ for any trinity matrix, by the properties $a_1 = \pm \frac{1}{n} Trace(A)$.

The method of construction of $n^{th}$ order trinity matrices is to solve the system of nonhomogeneous quadratic equations:

$$a_1^2 + a_2^2 + a_3^2 + \ldots + a_n^2 = 1,$$
$$a_1 a_{k+1} + \ldots + a_{n-k} a_n + a_{n-k+1} a_1 + \ldots a_n a_k = 0 .$$

where $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$.

From property (6), we can find

$$a_1 = \pm \frac{1}{n} Trace(A) .$$

Then the nonhomogeneous system for each $n$ has limited trinity matrices, and if $n$ increases, the number of trinity matrices will increase. From property (7), we can find $a_1$ first. The following table listed some possible values of the diagonal entries of $a_1$. Part results are listed in the following table.

**Table 1 Part results.**

| $a_1(n)$ | $n=2k+1$ | $n=2k+2$ | |
|---|---|---|---|
| $k=1$ $n=3$ | $a_1=\mp\frac{1}{3}$ | $a_1=\mp\frac{1}{2}$ | $n=4$ |
| $k=2$ $n=5$ | $a_1=\mp\frac{1}{5};\mp\frac{3}{5}$ | $a_1=\mp\frac{1}{3};\mp\frac{2}{3},$ | $n=6$ |
| $k=3$ $n=7$ | $a_1=\mp\frac{1}{7};\mp\frac{3}{7};\mp\frac{5}{7};$ | $a_1=\mp\frac{1}{4};\mp\frac{2}{4},\mp\frac{3}{4}$ | $n=8$ |
| $k=4$ $n=9$ | $a_1=\mp\frac{1}{9};\mp\frac{3}{9};\mp\frac{5}{9};\mp\frac{7}{9};$ | $a_1=\mp\frac{1}{5};\mp\frac{2}{5},\mp\frac{3}{5},\mp\frac{4}{5}$ | $n=10$ |
| $k=5$ $n=11$ | $a_1=\mp\frac{1}{11};\mp\frac{3}{11};\mp\frac{5}{11};\mp\frac{7}{11};\mp\frac{9}{11}$ | $a_1=\mp\frac{1}{6};\mp\frac{2}{6},\mp\frac{3}{6},\mp\frac{4}{6};\mp\frac{5}{6}$ | $n=12$ |
| $k=6$ $n=13$ | $a_1=\mp\frac{1}{13};\mp\frac{3}{13};\mp\frac{5}{13};\mp\frac{7}{13};\mp\frac{9}{13};\mp\frac{11}{13}$ | $a_1=\mp\frac{1}{7};\mp\frac{2}{7},\mp\frac{3}{7},\mp\frac{4}{7};\mp\frac{5}{7};\mp\frac{6}{7}$ | $n=14$ |
| $k=7$ $n=15$ | $a_1=\mp\frac{1}{15};\mp\frac{3}{15};\mp\frac{5}{15};\mp\frac{7}{15};\mp\frac{9}{15};\mp\frac{11}{15};\mp\frac{13}{15}$ | $a_1=\mp\frac{1}{8};\mp\frac{2}{8},\mp\frac{3}{8},\mp\frac{4}{8};\mp\frac{5}{8};\mp\frac{6}{8};\mp\frac{7}{8}$ | $n=16$ |
| $k=9$ $n=17$ | $a_1=\mp\frac{1}{17};\mp\frac{3}{17};\mp\frac{5}{17};\mp\frac{7}{17};\mp\frac{9}{17};\mp\frac{11}{17};$ $\mp\frac{13}{17},\mp\frac{15}{17}$ | $a_1=\mp\frac{1}{9};\mp\frac{2}{9},\mp\frac{3}{9},\mp\frac{4}{9};\mp\frac{5}{9};\mp\frac{6}{9};\mp\frac{7}{9};$ $\mp\frac{8}{9};$ | $n=18$ |
| $n=2k-1$ | $a_1=\mp\frac{2i-1}{2k}$, where $i=1,2,3,\dots,k$ | $a_1=\mp\frac{i}{k}$, where $i=1,2,3,\dots,k-1$ | $n=2k$ |

## 4. Construction Even Order Trinity Matrices

Since $n$ is even, $A$ is symmetry, then we have

$$a_i = a_{2k+2-i}\ ,\quad i = 2,3,\dots,k$$

Or $i = k+1, k+2, k+3,\dots,2k.$

Where $a_1$ can be determined by the trace of function $p_n(\lambda)=\left|\lambda I - A_{n,n}\right|$ see the table above, then we have the following results.

The following examples show the procedure of construction of trinity matricesof $A_{n,n}$ when

$$n = 2k\ ,\quad k = 2,3,4,5,\dots$$

**For n = 4**, the characteristic polynomial $p(\lambda)=(\lambda-1)^i(\lambda+1)^{4-i}$ $i=1,2,3$ then, we have three cases:

$p_1(\lambda)=(\lambda-1)(\lambda+1)^3=\lambda^4+2\lambda^3-2\lambda-1$,

$a_1=-\frac{2}{4}=-\frac{1}{2}$;

$p_2(\lambda)=(\lambda-1)^2(\lambda+1)^2=\lambda^4-2\lambda^2+1$,

$a_1=0$, discarded

$p_3(\lambda)=(\lambda-1)^3(\lambda+1)=\lambda^4-2\lambda^3+2\lambda-1$,

$a_1=\frac{1}{2}$,

We have the following 4 trinity matrices:

$$A=\pm\frac{1}{2}curculant(-1 \quad 1 \quad 1 \quad 1)$$

$$A=\pm\frac{1}{2}curculant(1 \quad 1 \quad -1 \quad 1).$$

**For n = 6:**, the characteristic polynomial is $p(\lambda)=(\lambda-1)^i(\lambda+1)^{6-i}$ the possible $a_1=\pm\frac{1}{3}, \quad \pm\frac{2}{3}$, then we find

$$A=\pm\frac{1}{3}curculant(-2 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1)$$

$$A=\pm\frac{1}{3}curculant(1 \quad 1 \quad 1 \quad -2 \quad 1 \quad 1)$$

$$A=\pm\frac{1}{3}curculant(-2 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1)$$

$$A=\pm\frac{1}{3}curculant(-1 \quad 1 \quad -1 \quad -2 \quad -1 \quad 1);$$

**For n = 8:**

$$A=\pm\frac{1}{4}curculant(-3 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1)$$

$$A=\pm\frac{1}{4}curculant(1 \quad 1 \quad 1 \quad 1 \quad -3 \quad 1 \quad 1 \quad 1)$$

$$A=\pm\frac{1}{4}curculant(-3 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1)$$

$$A=\pm\frac{1}{4}curculant(1 \quad -1 \quad 1 \quad -1 \quad -3 \quad -1 \quad 1 \quad -1);$$

**For n = 10:**

$$A=\pm\frac{1}{5}curculant(-4 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1);$$

$$A=\pm\frac{1}{5}curculant(1 \quad 1 \quad 1 \quad 1 \quad 1 \quad -4 \quad 1 \quad 1 \quad 1 \quad 1);$$

$$A=\pm\frac{1}{5}curculant(-4 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1);$$

$$A=\pm\frac{1}{5}curculant(-1 \quad 1 \quad -1 \quad 1 \quad -1 \quad -4 \quad -1 \quad 1 \quad -1 \quad 1);$$

$$A=\pm\frac{1}{10}curculant(-4 \quad 4 \quad 1 \quad 4 \quad 1 \quad 4 \quad 1 \quad 4 \quad 1 \quad 4)$$

$$A=\pm\frac{1}{10}curculant(-4 \quad -1 \quad -4 \quad -1 \quad -4 \quad 4 \quad -4 \quad -1 \quad -4 \quad -1);$$

**For n = 12:**

$$A=\pm\frac{1}{6}curculant(-5 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1);$$

$$A = \pm\frac{1}{6}curculant(-5 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1);$$

$$A = \pm\frac{1}{2\sqrt{5}}curculant(3 \quad 1 \quad -1 \quad 1 \quad -1 \quad 1 \quad 1 \quad 1 \quad -1 \quad 1 \quad -1 \quad 1);$$

$$A = \pm\frac{1}{\sqrt{14}}curculant(2 \quad 1 \quad -1 \quad 1 \quad -1 \quad 1 \quad 2 \quad 1 \quad -1 \quad 1 \quad -1 \quad 1);$$

$$A = \pm\frac{1}{\sqrt{15}}curculant(2 \quad -1 \quad -1 \quad 1 \quad 1 \quad 1 \quad -1 \quad 1 \quad 1 \quad 1 \quad -1 \quad 1);$$

$$A = \pm\frac{1}{6}curculant(1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad -5 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1);$$

$$A = \pm\frac{1}{6}curculant(1 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1 \quad -5 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1);$$

$$A = \pm\frac{1}{2\sqrt{3}}curculant(1 \quad -1 \quad -1 \quad 1 \quad 1 \quad 1 \quad -1 \quad 1 \quad 1 \quad 1 \quad -1 \quad -1);$$

$$A = \pm\frac{1}{2\sqrt{5}}curculant(1 \quad 1 \quad -1 \quad 1 \quad -1 \quad 1 \quad 3 \quad 1 \quad -1 \quad 1 \quad -1 \quad 1);$$

**For n = 14:**

$$A = \pm\frac{1}{7}curculant(-6 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1);$$

$$A = \pm\frac{1}{7}curculant(1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad -6 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1);$$

$$A = \pm\frac{1}{7}curculant(-6 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1);$$

$$A = \pm\frac{1}{7}curculant(-1 \quad 1 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1 \quad -6 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1 \quad 1);$$

**For n = 16:**

$$A = \pm\frac{1}{8}curculant(-7 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1);$$

$$A = \pm\frac{1}{8}curculant(-7 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1);$$

**For n = 18:**

$$A = \pm\frac{1}{9}curculant(-8 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1);$$

$$A = \pm\frac{1}{9}curculant(-8 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1);$$

In general, for $n = 2k$

$$A = \pm\frac{1}{k}curculant(x_1, x_2, x_3, \ldots.x_k, x_{k+1}, x_{k+2}, x_{k+3}, \ldots x_{n-1}, x_n)$$

$$= \pm \frac{1}{k} curculant(1-k, x_2, x_3, x_4 ....x_k, 1, x_k, x_{k-1}, ..x_4, x_3, x_2)$$

Where $x_2 = x_{2k} = -1, x_3 = x_{2k-1} = 1, ...x_{k+1} = 1 \quad for \quad k = 2,4,6,8,....$

$x_2 = x_{2k} = -1, x_3 = x_{2k-1} = 1, ...x_{k+1} = -1 \quad for \quad k = 3,5,7,9,....$

**Proposition 1** If n is an even integer, let $n = 2k, k \geq 4$ then the Trinity Matrices of order $n$ have $k-1$ pairs of non-trivial solutions

$$A = \pm \frac{i}{k-1} B, \quad A = \pm \frac{i}{k-1} C \quad i = 1,2,...,k-2$$

Where

$$B = circulant(1-k, 1,1,1,1,...1,1)$$

$$C = circulant(1-k, -1,1,-1,1,...1,-1)$$

## 5. Construction of Odd Order Trinity Matrices

If n is odd, we write $n = 2k+1, k = 2,3,...$ , by the symmetry of matrices gives $a_n = a_2, a_{n-1} = a_3, a_{n-2} = a_4 ..., a_{k+2} = a_{k+1}$,

For example, a nine by nine trinity matrix looks like the following matrix

$$A = Circulant(a_1, a_2, a_3, ..., a_n) = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_5 & a_4 & a_3 & a_2 \\ a_2 & a_1 & a_2 & a_3 & a_4 & a_5 & a_5 & a_4 & a_3 \\ a_3 & a_2 & a_1 & a_2 & a_3 & a_4 & a_5 & a_5 & a_4 \\ a_4 & a_3 & a_2 & a_1 & a_2 & a_3 & a_4 & a_5 & a_5 \\ a_5 & a_4 & a_3 & a_2 & a_1 & a_2 & a_3 & a_4 & a_5 \\ a_5 & a_5 & a_4 & a_3 & a_2 & a_1 & a_2 & a_3 & a_4 \\ a_4 & a_5 & a_5 & a_4 & a_3 & a_2 & a_1 & a_2 & a_3 \\ a_3 & a_4 & a_5 & a_5 & a_4 & a_3 & a_2 & a_1 & a_2 \\ a_2 & a_3 & a_4 & a_5 & a_5 & a_4 & a_3 & a_2 & a_1 \end{pmatrix}$$

The normal equation becomes

$$a_1^2 + 2a_2^2 + 2a_3^2 + ... + 2a_k^2 + 2a_{k+1}^2 = 1$$
$$a_1 + 2a + 2a_3 + ... + 2a_k + 2a_{k+1} = 1$$

With $\left\lfloor \frac{n}{2} \right\rfloor = k$ , *orthogonal equations* can be rewritten accordingly. Hence the governing equations form a $(k+1) \times (k+1)$ system.

If $n = 2k+1, k = 2,3,\ldots$, noticed that $a_n = a_2, a_{n-1} = a_3, a_{n-2} = a_4 \ldots, a_{k+2} = a_{k+1}$, then we have

$$a_1^2 + 2a_2^2 + 2a_3^2 + \ldots + 2a_{k+1}^2 = 1, or$$
$$a_1 + 2a_2 + 2a_3 + \ldots + 2a_{k+1} = 1$$

The following are some examples of odd order Trinity Matrices:

**For n = 3**,

$$A = \pm\frac{1}{3} circulant(-1 \quad 2 \quad 2);$$

**For n = 5,** we find

$$A = \pm\frac{1}{3} circulant(-1 \quad -2 \quad 2 \quad 2 \quad -2)$$
$$A = \pm\frac{1}{3} circulant(-1 \quad 2 \quad -2 \quad -2 \quad 2)$$
$$A = \pm\frac{1}{5} circulant(-3 \quad 2 \quad 2 \quad 2 \quad 2)$$
$$A = \pm\frac{1}{5} circulant(1 \quad 1+\sqrt{5} \quad 1-\sqrt{5} \quad 1-\sqrt{5} \quad 1+\sqrt{5})$$
$$A = \pm\frac{1}{5} circulant(1 \quad 1-\sqrt{5} \quad 1+\sqrt{5} \quad 1+\sqrt{5} \quad 1-\sqrt{5})$$

**For n = 7**

$$A = \pm\frac{1}{5} circulant(-1, \quad -2, \quad 2, \quad 2, \quad 2, \quad 2, \quad -2)$$
$$A = \pm\frac{1}{\sqrt{89}} circulant(-1, \quad -6, \quad 2, \quad 2, \quad 2, \quad 2, \quad -6)$$
$$A = \pm\frac{1}{5} circulant(-1, \quad 2, \quad 2, \quad -2, \quad -2, \quad 2, \quad 2)$$
$$A = \pm\frac{1}{\sqrt{61}} circulant(-1, \quad 5, \quad 1, \quad -2, \quad -2, \quad 1, \quad 5)$$
$$A = \pm\frac{1}{3\sqrt{3}} circulant(-3, \quad 2, \quad 1, \quad 2, \quad 2, \quad 1, \quad 2)$$
$$A = \pm\frac{1}{\sqrt{33}} circulant(-3, \quad -2, \quad 2, \quad -2 \quad -2, \quad 2, \quad -2)$$
$$A = \pm\frac{1}{\sqrt{97}} circulant(-3, \quad 6, \quad 2, \quad 2 \quad 2, \quad 2, \quad 6)$$
$$A = \pm\frac{1}{7} circulant(-5, \quad 2, \quad 2, \quad 2 \quad 2, \quad 2, \quad 2)$$
$$A = \pm\frac{1}{7} circulant(-5, \quad -2, \quad 3, \quad -2 \quad -2, \quad 3, \quad -2)$$

$$A = \pm \frac{1}{\sqrt{113}} circulant(-5, \quad 2, \quad 2, \quad -6 \quad -6, \quad 2, \quad 2)$$

$$A = \pm \frac{1}{\sqrt{209}} circulant(-11, \quad -2, \quad 6, \quad -2 \quad -2, \quad 6, \quad -2)$$

$$A = \pm \frac{1}{\sqrt{377}} circulant(-17, \quad 2, \quad 2, \quad 6 \quad 6, \quad 2, \quad 2)$$

$$A = \pm \frac{1}{\sqrt{850}} circulant(-28, \quad 1, \quad 4, \quad 4, \quad 4, \quad 4, \quad 1)$$

$$A = \pm \frac{1}{\sqrt{210}} circulant(12, \quad 1, \quad -4, \quad 4, \quad 4, \quad -4, \quad 1)$$

**For n = 9** , we find

$$A = \pm \frac{1}{\sqrt{33}} circulant(-1 \quad -2 \quad 2 \quad 2 \quad -2 \quad -2 \quad 2 \quad 2 \quad -2)$$

$$A = \pm \frac{1}{\sqrt{33}} circulant(-1 \quad 2 \quad -2 \quad 2 \quad 2 \quad 2 \quad 2 \quad -2 \quad 2)$$

$$A = \pm \frac{1}{\sqrt{41}} circulant(-3 \quad 2 \quad -2 \quad -2 \quad -2 \quad -2 \quad -2 \quad -2 \quad 2)$$

$$A = \pm \frac{1}{\sqrt{41}} circulant(-3 \quad 2 \quad 2 \quad 2 \quad -2 \quad -2 \quad 2 \quad 2 \quad 2)$$

$$A = \pm \frac{1}{\sqrt{41}} circulant(-3 \quad 2 \quad 2 \quad -2 \quad -2 \quad -2 \quad -2 \quad 2 \quad 2)$$

$$A = \pm \frac{1}{\sqrt{57}} circulant(-5 \quad -2 \quad 2 \quad -2 \quad 2 \quad 2 \quad -2 \quad 2 \quad -2)$$

$$A = \pm \frac{9}{7} circulant(-7 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2)$$

$$A = \pm \frac{1}{21} \left(-11, \quad -8, \quad 4, \quad -8, \quad 4, \quad 4, \quad -8, \quad 4, \quad -8\right)$$

$$A = \pm \frac{1}{6} \left(-4, \quad -1, \quad 2, \quad -1, \quad 2, \quad 2, \quad -1, \quad 2, \quad -1\right)$$

$$A = \pm \frac{1}{9} \left(1, \quad -2, \quad 4, \quad 4, \quad -2, \quad -2, \quad 4, \quad 4, \quad -2\right)$$

$$A = \pm \frac{1}{9} \left(1, \quad 4 \quad -2 \quad -2 \quad 4 \quad 4 \quad -2 \quad -2 \quad 4\right)$$

**For n = 11**

$$A = \pm \frac{1}{7} circulant(-1 \quad 2 \quad 2 \quad 2 \quad -2 \quad 2 \quad 2 \quad -2 \quad 2 \quad 2 \quad 2)$$

$$A = \pm \frac{1}{7} circulant(-1 \quad 2 \quad 2 \quad -2 \quad 2 \quad 2 \quad 2 \quad 2 \quad -2 \quad 2 \quad 2)$$

$$A = \pm\frac{1}{7} circulant(-1 \quad 2 \quad -2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad -2 \quad 2)$$

$$A = \pm\frac{1}{7} circulant(-1 \quad 2 \quad 2 \quad 2 \quad -2 \quad 2 \quad 2 \quad -2 \quad 2 \quad 2 \quad 2)$$

$$A = \pm\frac{1}{\sqrt{41}} circulant(-1 \quad 2 \quad -2 \quad -2 \quad -2 \quad -2 \quad -2 \quad -2 \quad -2 \quad -2 \quad 2)$$

$$A = \pm\frac{1}{4\sqrt{2}} circulant(-1 \quad 2 \quad -2 \quad -2 \quad -2 \quad 2 \quad 2 \quad -2 \quad -2 \quad -2 \quad 2)$$

$$A = \pm\frac{1}{7} circulant(-3 \quad -2 \quad -2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad -2 \quad -2)$$

$$A = \pm\frac{1}{7} circulant(-3 \quad -2 \quad -2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad -2 \quad -2)$$

$$A = \pm\frac{1}{\sqrt{89}} circulant(-7 \quad 2 \quad 2 \quad 2 \quad 2 \quad -2 \quad -2 \quad 2 \quad 2 \quad 2 \quad 2)$$

$$A = \pm\frac{1}{\sqrt{65}} circulant(-5 \quad 2 \quad -2 \quad -2 \quad -2 \quad -2 \quad -2 \quad -2 \quad -2 \quad -2 \quad 2)$$

$$A = \pm\frac{1}{\sqrt{65}} circulant(-5 \quad 2 \quad 2 \quad -2 \quad -2 \quad -2 \quad -2 \quad -2 \quad -2 \quad 2 \quad 2)$$

$$A = \pm\frac{1}{\sqrt{65}} circulant(-5 \quad 2 \quad 2 \quad 2 \quad -2 \quad -2 \quad -2 \quad -2 \quad 2 \quad 2 \quad 2)$$

$$A = \pm\frac{1}{\sqrt{65}} circulant(-5 \quad 2 \quad 2 \quad -2 \quad -2 \quad -2 \quad -2 \quad -2 \quad -2 \quad 2 \quad 2)$$

$$A = \pm\frac{1}{\sqrt{65}} circulant(-5 \quad 2 \quad -2 \quad -2 \quad -2 \quad -2 \quad -2 \quad -2 \quad -2 \quad -2 \quad 2)$$

$$A = \pm\frac{1}{\sqrt{65}} circulant(-5 \quad 2 \quad -2 \quad -2 \quad -2 \quad -2 \quad -2 \quad -2 \quad -2 \quad -2 \quad 2)$$

$$A = \pm\frac{1}{11} circulant(-9 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2)$$

**For n = 13**

$$A = \pm\frac{1}{\sqrt{57}} circulant(-3 \quad -2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad -2)$$

$$A = \pm\frac{1}{\sqrt{57}} circulant(-3 \quad 2 \quad -2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad -2 \quad 2)$$

$$A = \pm\frac{1}{\sqrt{57}} circulant(-3 \quad 2 \quad 2 \quad -2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad -2 \quad 2 \quad 2)$$

$$A = \pm\frac{1}{\sqrt{57}} circulant(-3 \quad 2 \quad 2 \quad 2 \quad -2 \quad 2 \quad 2 \quad 2 \quad 2 \quad -2 \quad 2 \quad 2 \quad 2)$$

$$A = \pm\frac{1}{\sqrt{57}} circulant(-3 \quad 2 \quad 2 \quad 2 \quad 2 \quad -2 \quad 2 \quad 2 \quad -2 \quad 2 \quad 2 \quad 2 \quad 2)$$

$$A = \pm\frac{1}{\sqrt{97}}circulant(-7 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad -2 \quad -2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2)$$

$$A = \pm\frac{1}{\sqrt{97}}circulant(-7 \quad 2 \quad -2 \quad 2 \quad -2 \quad 2 \quad -2 \quad -2 \quad 2 \quad -2 \quad 2 \quad -2 \quad 2)$$

$$A = \pm\frac{1}{\sqrt{97}}circulant(-7 \quad 2 \quad -2 \quad -2 \quad -2 \quad -2 \quad -2 \quad -2 \quad -2 \quad -2 \quad -2 \quad -2 \quad 2)$$

$$A = \pm\frac{1}{\sqrt{129}}circulant(-9 \quad -2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad -2)$$

$$A = \pm\frac{1}{\sqrt{129}}circulant(-9 \quad -2 \quad 2 \quad -2 \quad 2 \quad -2 \quad 2 \quad 2 \quad -2 \quad 2 \quad -2 \quad 2 \quad -2)$$

$$A = \pm\frac{1}{\sqrt{129}}circulant(-9 \quad 2 \quad -2 \quad 2 \quad -2 \quad 2 \quad -2 \quad -2 \quad 2 \quad -2 \quad 2 \quad -2 \quad 2)$$

$$A = \pm\frac{1}{13}circulant(-11 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2)$$

$$A = \pm\frac{1}{13}circulant(-11 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2)$$

**For n = 15**

$$A = \pm\frac{1}{15}circulant(-13 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2)$$

$$A = \pm\frac{1}{\sqrt{1743}}circulant(-13 \quad 14 \quad 1 \quad 14 \quad 1 \quad 14 \quad 1 \quad 14 \quad 14 \quad 1 \quad 14 \quad 1 \quad 14 \quad 1 \quad 14)$$

$$A = \pm\frac{1}{\sqrt{171}}circulant(-11 \quad 2 \quad -2 \quad 2 \quad -2 \quad 2 \quad -2 \quad 2 \quad 2 \quad -2 \quad 2 \quad -2 \quad 2 \quad -2 \quad 2)$$

$$A = \pm\frac{1}{\sqrt{171}}circulant(-11 \quad -2 \quad 2 \quad -2 \quad 2 \quad -2 \quad 2 \quad -2 \quad -2 \quad 2 \quad -2 \quad 2 \quad -2 \quad 2 \quad -2)$$

$$A = \pm\frac{1}{\sqrt{31}}circulant(-5 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1 \quad 1 \quad 1)$$

$$A = \pm\frac{1}{\sqrt{31}}circulant(-5 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1 \quad 1 \quad 1)$$

$$A = \pm\frac{1}{\sqrt{57}}circulant(-1 \quad 2 \quad -2 \quad 2 \quad 2 \quad 2 \quad -2 \quad -2 \quad -2 \quad -2 \quad 2 \quad 2 \quad 2 \quad -2 \quad 2)$$

$$A = \pm\frac{1}{15\sqrt{7}}circulant(-1 \quad 14 \quad -1 \quad 14 \quad -1 \quad 14 \quad -1 \quad 14 \quad 14 \quad -1 \quad 14 \quad -1 \quad 14 \quad -1 \quad 14)$$

## 6. Geometry Approach

**Definition:** The intersection of a unit sphere $a_1^2 + a_2^2 + a_3^2 + ... + a_n^2 = 1$ and a plane $a_1 + a_2 + a_3 + ... + a_n = 1$ in $R^n$ is

$$\begin{cases} a_1^2 + a_2^2 + a_3^2 + ... + a_n^2 = 1 & (1) \\ a_1 + a_2 + a_3 + ... + a_n = 1 & (2) \end{cases}$$

We find if $n = 5$

$$(\frac{1}{n})^2 + (\frac{1+\sqrt{n}}{n})^2 + (\frac{1-\sqrt{n}}{n})^2 + (\frac{1-\sqrt{n}}{n})^2 + (\frac{1+\sqrt{n}}{n})^2 = 1$$

$$\frac{1}{n} + \frac{1-\sqrt{n}}{n} + \frac{1+\sqrt{n}}{n} + \frac{1+\sqrt{n}}{n} + \frac{1-\sqrt{n}}{n} = 1$$

Hence we find

$$A = \pm\frac{1}{5} circulant(1 \quad 1+\sqrt{5} \quad 1-\sqrt{5} \quad 1-\sqrt{5} \quad 1+\sqrt{5})$$

$$A = \pm\frac{1}{5} circulant(1 \quad 1-\sqrt{5} \quad 1+\sqrt{5} \quad 1+\sqrt{5} \quad 1-\sqrt{5})$$

These two $5$ by 5 circulant matrices are Trinity Matrices. We can generalize this pattern to any integer $n = 4k + 1$ where

$$A = \pm\frac{1}{n} circulant(1 \quad 1+\sqrt{n} \quad 1-\sqrt{n} \quad 1-\sqrt{n} \quad 1+\sqrt{n}; ... \quad 1+\sqrt{n}, \quad 1-\sqrt{n} \quad 1-\sqrt{n} \quad 1+\sqrt{n});$$ Or

$$A = \pm\frac{1}{n} circulant(1 \quad 1-\sqrt{n} \quad 1+\sqrt{n} \quad 1+\sqrt{n} \quad 1-\sqrt{n}; ... \quad 1-\sqrt{n}, \quad 1+\sqrt{n} \quad 1+\sqrt{n} \quad 1-\sqrt{n});$$

$$k = 1,2,3,4,5,...$$

The following results are some examples:

$n = 5$

$$A = \pm\frac{1}{5} circulant(1 \quad 1+\sqrt{5} \quad 1-\sqrt{5} \quad 1-\sqrt{5} \quad 1+\sqrt{5})$$

$$A = \pm\frac{1}{5} circulant(1 \quad 1-\sqrt{5} \quad 1+\sqrt{5} \quad 1+\sqrt{5} \quad 1-\sqrt{5})$$

$n = 9$

$$A = \pm\frac{1}{9}\left(1, \quad 1-\sqrt{9}, \quad 1+\sqrt{9} \quad 1+\sqrt{9}, \quad 1-\sqrt{9}, \quad 1-\sqrt{9}, \quad 1+\sqrt{9}, \quad 1+\sqrt{9}, \quad 1-\sqrt{9}\right)$$

$$A = \pm\frac{1}{9}\left(1, \quad 1+\sqrt{9} \quad 1+\sqrt{9} \quad 1+\sqrt{9}, \quad 1+\sqrt{9} \quad 1+\sqrt{9} \quad 1+\sqrt{9}, \quad 1+\sqrt{9}, \quad 1+\sqrt{9}\right)$$

Or

$$A = \pm\frac{1}{9}\left(1, \quad -2, \quad 4, \quad 4, \quad -2, \quad -2, \quad 4, \quad 4, \quad -2\right)$$

$$A = \pm\frac{1}{9}\left(1, \quad 4 \quad -2 \quad -2 \quad 4 \quad 4 \quad -2 \quad -2 \quad 4\right)$$

$n = 13$ , $p = 1+\sqrt{13}$, $q = 1-\sqrt{13}$,

$$A = \pm\frac{1}{13}\left(1 \quad p \quad q \quad q \quad p; \quad p \quad q \quad q, \quad p; \quad p \quad q \quad q \quad p\right)$$

$$A = \pm\frac{1}{13}\left(1 \quad q \quad p \quad p \quad q; \quad q \quad p \quad p, \quad q; \quad q \quad p \quad p \quad q\right)$$

$n = 17$ , $p = 1+\sqrt{17}$, $q = 1-\sqrt{17}$,

$$A = \pm\frac{1}{17} circulant(1, \quad p \quad q \quad q \quad p, \quad p \quad q \quad q \quad p, \quad p \quad q \quad q \quad p; \quad p \quad q \quad q \quad p \quad );$$

$$A = \pm\frac{1}{17} circulant(1, \quad q \quad p \quad p \quad q, \quad q \quad p \quad p \quad q, \quad q \quad p \quad p \quad q, \quad q \quad p \quad p \quad q);$$

$n = 21$ , $p = 1+\sqrt{21}$, $q = 1-\sqrt{21}$, then

$$A = \pm\frac{1}{21} circulant$$
$$(1, \quad p \quad q \quad q \quad p, \quad p \quad q \quad q \quad p, \quad p \quad q \quad q \quad p, \quad p \quad q \quad q \quad p, \quad p \quad q \quad q \quad p);$$

$$A = \pm\frac{1}{21} circulant$$
$$(1, \quad q \quad p \quad p \quad q, \quad q \quad p \quad p \quad q, \quad q \quad p \quad p \quad q, \quad q \quad p \quad p \quad q, \quad q \quad p \quad p \quad q);$$

$n = 25$ , $p = 1+\sqrt{25} = 6$, $q = 1-\sqrt{25} = -4$,

$$A = \pm\frac{1}{25} circulant$$
$$(1, \quad 6 \quad -4 \quad -4 \quad 6, \quad 6 \quad -4 \quad -4 \quad 6, \quad 6 \quad -4 \quad -4 \quad 6, \quad 6 \quad -4 \quad -4 \quad 6, \quad 6 \quad -4 \quad -4 \quad 6, \quad 6, \quad -4 \quad -4 \quad 6);$$

$$A = \pm\frac{1}{25} circulant$$
$$(1, \quad -4 \quad 6 \quad 6 \quad -4, \quad -4 \quad 6 \quad 6 \quad -4, \quad -4 \quad 6 \quad 6 \quad -4, \quad -4 \quad 6 \quad 6 \quad -4, \quad -4 \quad 6 \quad 6 \quad -4, \quad -4, \quad 6 \quad 6 \quad -4);$$

$n = 29$ , $p = 1+\sqrt{29}$ $q = 1-\sqrt{29}$, then

$$A = \pm\frac{1}{29} circulant$$
$$(1, \quad p \quad q \quad q \quad p, \quad p \quad q \quad q \quad p, \quad p \quad q \quad q \quad p, \quad p \quad q \quad q \quad p, \quad p \quad q \quad q \quad p, \quad p \quad q \quad q \quad p, \quad p \quad q \quad q \quad p);$$

$$A = \pm\frac{1}{29} circulant$$
$$(1, \quad q \quad p \quad p \quad q, \quad q \quad p \quad p \quad q, \quad q \quad p \quad p \quad q, \quad q \quad p \quad p \quad q, \quad q \quad p \quad p \quad q, \quad q \quad p \quad p \quad q, \quad q \quad p \quad p \quad q);$$

For any positive integer $n \geq 3$, n by n Trinity Matrices always exist, and the numbers of Trinity Matrices of any fixed rank $n \geq 3$ are limited. And when the rank of Trinity Matrices increases, it is difficult to solve the following nonhomogeneous quadratic system

$$a_1^2 + a_2^2 + a_3^2 + \ldots + a_n^2 = 1$$

$$a_1 a_{k+1} + \ldots + a_{n-k} a_n + a_{n-k+1} a_1 + \ldots a_n a_k = 0$$

where $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$, if $n$ is "large". Readers are welcome to develop or suggest more effective methods to construct Trinity Matrices, and find more applications.

## 7. Summary

The original idea to classify the solutions of twin jets was originated by Tam & Thies when they investigated "Instability of Rectangular Jets" in 1992 [2].

The Trinity Matrices were first used in the studying upstream propagated acoustic waves and Kelvin-Helmholtz instability waves of supersonic jets with N- engines [3]. We discovered and constructed TrinityMatrices such that we are able to classify the governing equations into 2N families, then derived 2N dispersion relations.

We discovered more applications of Trinity Matrices in other fields, and would like to publicize later.

## References

[1] T.W. Hungerford, Algebra, Springer-Verlag, New York, 1980, ISBN 0-387-90518-9, p. 121.

[2] C.K.W. Tam, A.T. Thies, Instability of rectangular jets, .Journal of Fluid Mechanics 248 (1993) 425-448.

[3] J.Z. DU, L. WANG, Dispersion relations for supersonic multiple virtual jets, Discrete and Continues Dynamical Systems (2011 Supplement), pp. 381-390.