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# Use of Different Representations of Ceva's Theorem for Development of Geometric Properties of a Triangle

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**Abstract**: The paper describes different representations of Ceva's theorem and shows how the use of these theorems brings about surprising and wide-ranging results both in the study of geometric properties of a triangle and in geometrical constructions using a straightedge only.

**Key words**: Ceva's theorem, cevians, triangle geometry.

## 1. Introduction

Theorems in geometry, in reality, in the program of studies in many cases, do not encompass theorems from antiquity, such as Menelaus' and Ptolemy's theorems, or from the classical age, such as Ceva's and Pascal's theorems. Those theorems can enrich the knowledge of teachers and students, expand their mathematical toolbox, and mainly—give rise to geometrical properties and eventually, propose simple and short elegant solutions and results.

For example, Ceva's theorem has considerable didactic potential and it allows one to discover surprising properties in the triangle, thus enhancing the studies of geometry.

Ceva's theorem was originally published by the Italian mathematician Giovani Ceva in 1678 (see for example [1-3]). This theorem plays an important role in Euclidean geometry, and especially in the geometry of the triangle. Since Ceva's theorem concerns segments in the triangle which connect its vertices with the opposite sides, these segments are named cevians after Ceva. The theorem provides a sufficient and necessary condition for the three cevians to meet at the same point (such cevians are called concurrent cevians). Ceva's theorem is related and constitutes a link to many other theorems in geometry. For instance, using this theorem one can immediately conclude that the three medians of a triangle meet at one point. A similar result can be obtained also for the three altitudes and for the three angle bisectors of a triangle.

Ceva's theorem has many different proofs and several different representations. In the paper we will consider the following representations: (1) The classical representation; (2) Representation by Van Aubel's theorem; (3) Representation by ratios of areas; (4) Two trigonometric representations [4].

These representations result in surprising and interesting results. Some of them are given in the paper.

## 2. Different Representations of Ceva's Theorem

### *2.1 The Classical Representation*

Three concurrent cevians *AD*, *BE*, *CF* which intersect at the same point *O* are given in the triangle *ABC* (see Figure 1).

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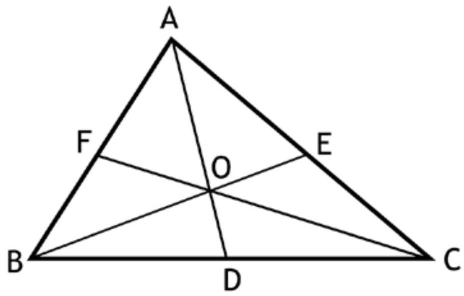


**Fig. 1**

Hence,

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}=1 \qquad \text{(I)}$$

One of the known proofs of Ceva's theorem is by means of Menelaus' theorem.

Menelaus' theorem states that for the triangle *ABD* that is intersected by a straight line passing through the points *C*, *O*, *F*, there holds:

$\frac{BC}{DC}\cdot\frac{DO}{OA}\cdot\frac{AF}{FB}=1$, therefore,

$$\frac{AO}{DO}=\frac{BC}{DC}\cdot\frac{AF}{FB} \qquad (1)$$

By using Menelaus' theorem again for the triangle *ADC* that is intersected by a straight line passing through the points *E*, *O*, *B*, we obtain:

$\frac{BC}{BD}\cdot\frac{DO}{OA}\cdot\frac{AE}{EC}=1$, therefore,

$$\frac{AO}{DO}=\frac{BC}{BD}\cdot\frac{AE}{EC} \qquad (2)$$

From relations (1) and (2) we obtain that $\frac{BC\cdot AF}{DC\cdot FB}=\frac{BC\cdot AE}{BD\cdot EC}$, and therefore,

$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}=1$ (see [1, pp. 4-5], [2, p. 137]).

**Note 1:**

As we have shown, Ceva's theorem follows simply and directly from Menelaus' theorem. It is therefore surprising that some 1,700 years have passed between discoveries of these two theorems.

*2.2 Representation by Van Aubel's Theorem*

From relations (1) and (2) it can be easily obtained that $\frac{AO}{DO}\cdot\left(\frac{DC}{BC}+\frac{BD}{BC}\right)=\frac{AF}{FB}+\frac{AE}{EC}$.

Since $\frac{DC}{BC}+\frac{BD}{BC}=\frac{DC+BD}{BC}=\frac{BC}{BC}=1$, we obtain that $\frac{AO}{DO}=\frac{AF}{FB}+\frac{AE}{EC}$ (II)

This is a famous theorem by Van Aubel (see for example [5]) that claims that if three cevians *AD*, *BE*, *CF* intersect at the point *O*, then (II) holds. The Van Aubel theorem is an alternative representation of Ceva's theorem, where the classical representation has a multiplicative nature and the representation by Van Aubel's theorem has and additive nature.

*2.3 Representation by Ratios of Areas*

We give a proof to Ceva's theorem by considering the areas of the triangles *AOB*, *AOC*, *BOC*, which we shall denote by $S_{AOB}$, $S_{AOC}$, and $S_{BOC}$ respectively).

The following relations hold for the areas:

$$\frac{S_{AOB}}{S_{AOC}}=\frac{S_{BOD}}{S_{DOC}}=\frac{BD}{DC} \qquad (3)$$

$$\frac{S_{BOC}}{S_{AOB}}=\frac{S_{OEC}}{S_{OEA}}=\frac{CE}{AE} \qquad (4)$$

$$\frac{S_{AOC}}{S_{BOC}}=\frac{S_{AOF}}{S_{BOF}}=\frac{AF}{FB} \qquad (5)$$

We multiply the relations (3), (4) and (5) and obtain that $1=\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}$. Hence it follows that the three cevians passing through the point $O$ divide the triangle into 6 triangles with areas of $S_1, S_2, \ldots, S_6$ (see Figure 2), so that $\frac{S_1 \cdot S_3 \cdot S_5}{S_2 \cdot S_4 \cdot S_6} = 1$ (III)

This last relation is an alternative representation of Ceva's theorem using area ratios (see [1, p. 5]).

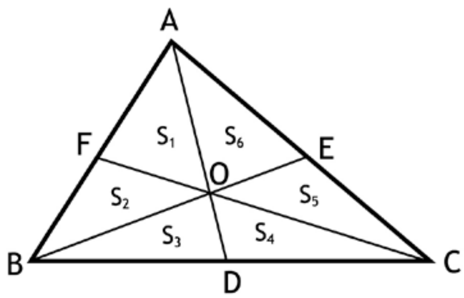


**Fig. 2**

*2.4 Trigonometric Representations*

**2.4.1 The first trigonometric representation**

It can be seen that

$$\frac{S_{ABD}}{S_{ADC}} = \frac{AB \sin A_1}{AC \sin A_2} = \frac{BD}{DC} \quad (6)$$

$$\frac{S_{BCE}}{S_{BAE}} = \frac{BC \sin B_1}{AB \sin B_2} = \frac{CE}{EA} \quad (7)$$

$$\frac{S_{CAF}}{S_{CBF}} = \frac{AC \sin C_1}{BC \sin C_2} = \frac{AF}{FB} \quad (8)$$

We multiply the relations (6), (7) and (8) and obtain:

$$\frac{\sin A_1}{\sin B_2} \cdot \frac{\sin B_1}{\sin C_2} \cdot \frac{\sin C_1}{\sin A_2} = \frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} \quad (9).$$

By using the Law of Sines in the triangles $AOB$, $BOC$ and $COA$, it follows that $\frac{\sin A_1}{\sin B_2} = \frac{BO}{AO}$, $\frac{\sin B_1}{\sin C_2} = \frac{CO}{BO}$, $\frac{\sin C_1}{\sin A_2} = \frac{AO}{CO}$. It therefore also follows that the left-hand side in relation (9) is equal to 1, and therefore we conclude that the three cevians in the triangle $ABC$ that pass through the point O divide the angles of the triangle into pairs of angles $A_1$, $A_2$, $C_1$, $C_2$, $B_1$, $B_2$ (see Figure 3), such that

$$\sin A_{1.} \cdot \sin B_1 \cdot \sin C_1 = \sin A_{2.} \cdot \sin B_2 \cdot \sin C_2 \quad \text{(IV)}.$$

This is the alternative representation of Ceva's theorem using trigonometry.

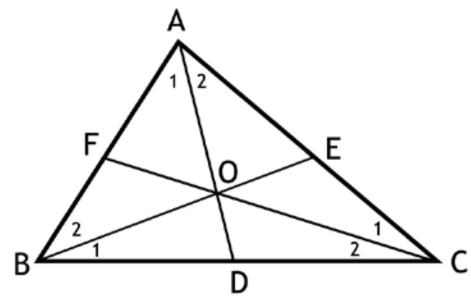


**Fig. 3**

**2.4.2 The second trigonometric representation**

The area ratios (see Figure 4) can be written as:

$$\frac{S_{ADF}}{S_{BDF}} = \frac{AD \sin D_2}{BD \sin D_1} = \frac{AF}{BF} \quad (10)$$

$$\frac{S_{CDE}}{S_{ADE}} = \frac{DC \sin D_4}{AD \sin D_3} = \frac{CE}{AE} \quad (11)$$

From relations (10) and (11) we obtain that

$$\frac{\sin D_2 \cdot \sin D_4}{\sin D_1 \cdot \sin D_3} = \frac{AF}{BF} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} \quad (12)$$

From relation (12) one can formulate the second trigonometric representation of Ceva's theorem as follows: the three cevians in the triangle $ABC$ that pass through the point $O$ divide the angle $D$ (see Figure 4) into four angles $D_1$, $D_2$, $D_3$, $D_4$, such that:

$$\frac{\sin D_2 \cdot \sin D_4}{\sin D_1 \cdot \sin D_3} = 1 \quad \text{(V)}$$

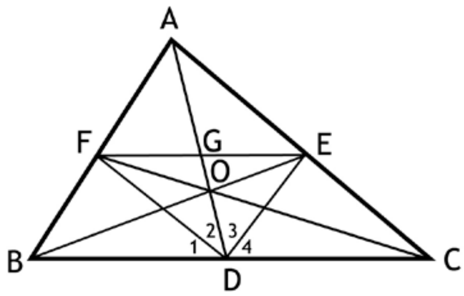


**Fig. 4**

## 3. Surprising Applications of Representations of Ceva's Theorem

*3.1 Proposition 1*

Three concurrent cevians divide the given triangle (whose area is S) into 6 triangles. It is clear that the area of at least one of them does not exceed S/6. What is surprising is that there is at least another triangle whose area also does not exceed S/6.

Proof:

$$\frac{S}{6}=\frac{\sum_{i=1}^{6}S_i}{6}\geq\sqrt[6]{\prod_{i=1}^{6}S_i}\ .$$

From (III), $\sqrt[6]{\prod_{i=1}^{6}S_i}=\sqrt[3]{S_1\cdot S_3\cdot S_5}=\sqrt[3]{S_2\cdot S_4\cdot S_6}$ .

Therefore $\frac{S}{6}\geq Min(S_1,S_3,S_5)$ and

$$\frac{S}{6}\geq Min(S_2,S_4,S_6)\ .$$

*3.2 Proposition 2*

Three concurrent cevians divide the angles of the triangle to produce 6 angles. It is clear that at least one of them does not exceed 30°, but it turns out that there is at least another angle whose value also does not exceed 30°.

Proof:

$$\begin{aligned}&\sin A_{1.}\cdot\sin A_2\\&=0.5(\cos(A_1-A_2)-\cos A)\\&=0.5(\cos(A_1-A_2)-1+2\sin^2(A/2))\leq\sin^2(A/2)\end{aligned}$$

Hence,

$$\begin{aligned}&\sin A_{1.}\cdot\sin B_1\cdot\sin C_1\cdot\sin A_{2.}\cdot\sin B_2\cdot\sin C_2\\&\leq\sin^2(A/2)\sin^2(B/2)\sin^2(C/2)\end{aligned}.$$

Therefore, from the representation (IV), we have:

$$\sin A_{1.}\cdot\sin B_1\cdot\sin C_1\leq\sin(A/2)\sin(B/2)\sin(C/2)\ .$$

It is known that the following inequality holds for any triangle:

$\sin(A/2)\sin(B/2)\sin(C/2)\leq 1/8$ (see for example [6]), therefore:

$\sin A_{1.}\cdot\sin B_1\cdot\sin C_1\leq 1/8$ and

$$Min(\sin A_{1.},\sin B_1,\sin C_1)\leq 1/2\ .$$

This suggests that at least one of the angles $A_1$, $B_1$, $C_1$ does not exceed 30°. In the same manner we obtain that at least one of the angles $A_2$, $B_2$, $C_2$ also does not exceed 30°.

*3.3 Proposition 3*

Consider the triangle *DEF*. It turns out that its area is at least 4 times smaller than the area of the triangle *ABC*.

We denote: BD/DC = $\alpha$; CE/EA = $\beta$; AF/FB = $\gamma$.

From the representation (I) we have $\alpha\beta\gamma = 1$, and there holds:

$$1+\alpha\geq 2\sqrt{\alpha}\quad,\quad 1+\beta\geq 2\sqrt{\beta}\quad,\quad 1+\gamma\geq 2\sqrt{\gamma}\quad,$$

therefore

$$(1+\alpha)(1+\beta)(1+\gamma)\geq 8\sqrt{\alpha\beta\gamma}=8 \qquad (13)$$

From Routh's theorem ([3, p. 382], [7]) there holds $S_{DEF}=\frac{S_{ABC}(1+\alpha\beta\gamma)}{(1+\alpha)(1+\beta)(1+\gamma)}$, and from this and from relation (13) we obtain that $S_{DEF}\le\frac{S_{ABC}}{4}$.

*3.4 Proposition 4*

Representation (II) yields two interesting inequalities:
$\frac{AO}{OD}+\frac{BO}{OE}+\frac{CO}{OF}\ge 6$ and $\frac{AO}{OD}\cdot\frac{BO}{OE}\cdot\frac{CO}{OF}\ge 8$ (see also [2, pp. 139-140], [8, pp. 114-115]).

Proof:

From the representation (II) there holds: $\frac{AO}{DO}=\frac{AF}{FB}+\frac{AE}{EC}$, in a similar manner we have $\frac{BO}{OE}=\frac{BD}{DC}+\frac{BF}{FA}$ and $\frac{CO}{OF}=\frac{CE}{EA}+\frac{CD}{DB}$, and by adding the three expressions together with obtain:

$$\frac{AO}{OD}+\frac{BO}{OE}+\frac{CO}{OF}$$
$$=\left(\frac{AF}{FB}+\frac{BF}{AF}\right)+\left(\frac{BD}{DC}+\frac{DC}{BD}\right)+\left(\frac{CE}{EA}+\frac{EA}{CE}\right)$$
$$\ge 2+2+2=6$$

where equality holds when the three cevians are the medians of the triangle.

Similarly we obtain:

$$\frac{AO}{DO}=\frac{AF}{FB}+\frac{AE}{EC}\ge 2\sqrt{\frac{AF}{FB}\cdot\frac{AE}{EC}},$$

$$\frac{BO}{OE}=\frac{BD}{DC}+\frac{BF}{FA}\ge 2\sqrt{\frac{BD}{DC}\cdot\frac{BF}{FA}}\text{, and}$$

$$\frac{CO}{OF}=\frac{CE}{EA}+\frac{CD}{DB}\ge 2\sqrt{\frac{CE}{EA}\cdot\frac{CD}{DB}}\text{, then}$$

$$\frac{AO}{OD}\cdot\frac{BO}{OE}\cdot\frac{CO}{OF}\ge 2\cdot 2\cdot 2=8$$

*3.5 Proposition 5*

Use of the representation (V) immediately suggests that if one cevian, for example *AD*, is the altitude in the triangle *ABC*, then for any two other cevians that meet on *AD*, there holds that the angle $D_1$ is equal to the angle $D_4$ (see Figure 5).

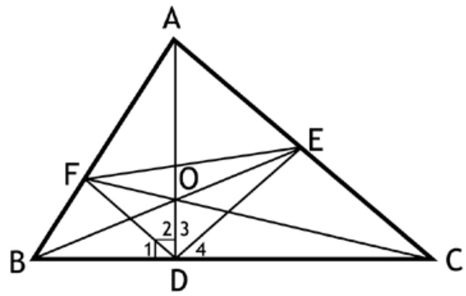


**Fig. 5**

Indeed, if $\angle$BDA=90$^{o}$ then $\sin D_1=\cos D_2$ and $\sin D_4=\cos D_3$. From the representation (V) we have

$$\frac{\sin D_2\cdot\sin D_4}{\sin D_1\cdot\sin D_3}=1 \quad\text{therefore}\quad \frac{\sin D_2\cdot\cos D_3}{\cos D_2\cdot\sin D_3}=1,$$

and therefore $tgD_2=tgD_3$ and $\angle D_2=\angle D_3$, $\angle D_1=\angle D_4$.

We note that this suggests that in the triangle known as the orthic triangle that is formed by the endpoints of the altitudes of the triangle *ABC*, these altitudes bisect its angles.

*3.6 Proposition 6*

From representation (III) we have that $\frac{S_1\cdot S_3\cdot S_5}{S_2\cdot S_4\cdot S_6}=1$. Hence follows an interesting fact that in order to calculate the areas $S_1$, $S_2$, … , $S_6$, it is enough to know only three of the areas. This proof requires complex algebra transformations and it will therefore not be shown.

We shall demonstrate this fact using a numeric example. Assuming that it is given that $S_1 = 3, S_2 = 7, S_3 = 4$, we calculate $S_4$, $S_5$, and $S_6$.

To this end we denote $S_4 = x, S_5 = y, S_6 = z$. Since there holds $\frac{S_1 + S_2}{S_3} = \frac{S_6 + S_5}{S_4} = \frac{AO}{OD}$, we have

$$\frac{10}{4} = \frac{y+z}{x}, \text{ or } y + z = \frac{5x}{2} \tag{14}$$

In a similar manner, $\frac{S_4 + S_3}{S_2} = \frac{S_6 + S_5}{S_1} = \frac{CO}{OF}$, and hence,

$$\frac{y+z}{3} = \frac{x+4}{7} \text{ and } y + z = \frac{3(x+4)}{7} \tag{15}$$

From relations (14) and (15) we find that $x = \frac{24}{29}$, and therefore $y + z = \frac{60}{29}$. From representation (III) one can write $3 \cdot 4 \cdot y = 7 \cdot \frac{24}{29} \cdot z$, and therefore $z = \frac{60}{43}, y = \frac{840}{1247}$.

*3.7 Proposition 7*

For the three concurrent cevians there holds: $\frac{AO}{OD} = 2 \cdot \frac{AG}{GD}$ (see Figure 4).

Proof:

We denote: BD/DC = α, CE/EA = β, AF/FB = γ. Then

$$\frac{AG}{GD} = \frac{S_{AFE}}{S_{DFE}} = \frac{\frac{S_{AFE}}{S_{ABC}}}{\frac{S_{DFE}}{S_{ABC}}} = \frac{\frac{AF \cdot AE}{AB \cdot AC}}{\frac{S_{DFE}}{S_{ABC}}} = \frac{\frac{\gamma}{\gamma+1} \cdot \frac{1}{\beta+1}}{\frac{S_{DFE}}{S_{ABC}}}.$$

Since $\frac{S_{DFE}}{S_{ABC}} = \frac{2}{(1+\alpha)(1+\beta)(1+\lambda)}$, we have

$$\frac{AG}{GD} = \frac{\gamma(\alpha+1)}{2} \tag{16}$$

From representation (II) we have $\frac{AO}{DO} = \gamma + \frac{1}{\beta}$, and therefore from (I),

$$\frac{AO}{DO} = \frac{\gamma\beta+1}{\beta} = \frac{\frac{1}{\alpha}+1}{\beta} = \frac{1+\alpha}{\alpha\beta} = \gamma(1+\alpha) \tag{17}$$

By comparing relations (16) and (17), we have $\frac{AO}{OD} = 2 \cdot \frac{AG}{GD}$.

## 4. Applications of Representations of Ceva's Theorem for Geometric Constructions using a Straightedge only

The classic geometrical constructions are limited to use of an unmarked straightedge and a compass. However, not all of these constructions can be performed using a straightedge only. All possible constructions using straightedge and compass can be built using a straightedge only, as long as any circle with its center is given (see for example [9]).

However if we do not have the circle with its center, the general problem of building using a straightedge only remains unsolved and requires a solution (if it generally exists) for each specific case.

Below we present two elegant constructions of these type, that are based on applications of some of representations of Ceva's theorem.

*4.1 Construction 1.*

From proposition 7 it can be easily deduced that if three points *A*, *O*, *D* lie on the same straight line, so that

$\frac{AO}{OD} = k$, then one can construct a point $G$ on the line that satisfies $\frac{AG}{GD} = \frac{k}{2}$ using a straightedge only.

So, let draw line $\ell$ through point $D$ (see Figure 6) and denote points $M$ and $N$ on line $\ell$ on both sides of point $D$. Now, we connect M and N with points $O$ and $A$ and denote the points of intersection of line $MO$ with line $AN$ by $P$ and of line $AM$ with line $NO$ by $Q$. Then, we draw line $PQ$ and denote the point of its intersection with $AD$ by $G$. From proposition 7 it follows that $\frac{AG}{GD} = \frac{AO}{2 \cdot OD} = \frac{k}{2}$.

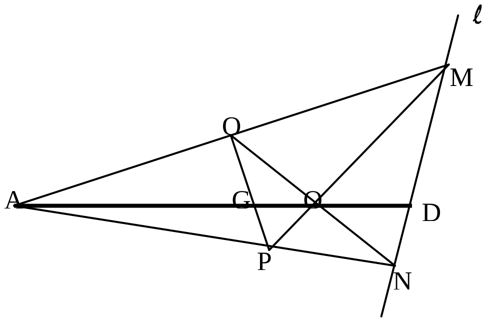


**Fig. 6**

*4.2 Construction 2*

On the plane, a line $\ell$ is given, as well as line m ("mirror"), that is perpendicular to $\ell$. Point $M$ is the point of intersection of $\ell$ and m. Ray s ("the ray of light") is the incident ray at point $M$. The problem is to construct the reflected ray r using a straightedge only (i.e. incident angle has to be equal to reflected angle).

For the construction (see Figure 7) let us denote two points $B$ and $C$ on $\ell$ on both sides of point $M$. Then we choose any point A (that does not coincide with M) on line m and draw lines $AB$ and $AC$. Let N be the point of intersection of ray s with $AB$. Now we draw line $NC$ and denote the point of its intersection with m by $O$, and the point of intersection of line $BO$ with $AC$ by $P$.

Due to proposition 5 we obtain that ∠AMN =∠AMP (see also [10]).

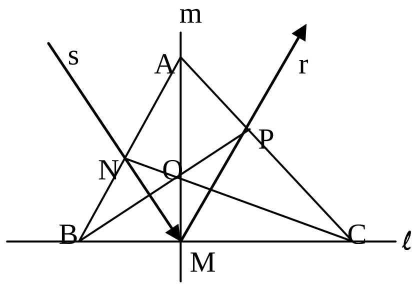


**Fig. 7**

And there are of course other applications of Ceva's theorem which result in interesting and surprising facts. We only presented some of them.

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