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Connectivity between Various Representations of Sets and their Relations among Teachers-Training Students

Amal Sharif-Rasslan[1] , Fathi Saleh[1] and Elias Abboud[1,2]

1. Department of Mathematics, The Academic Arab College for Education, Haifa 33145, Israel

2. Department of Mathematics, The Academic Arab College of Beit Berl, Kfar Saba 4490500, Israel

Abstract: The main goal of the research was to find out the connectivity between various representations of sets and attempt to clarify source of error when passing from one representation to another. A questionnaire which focused on the inclusion relation between sets, by switching this relation from the verbal, formal and verbal-logical to the visual representation, i.e., to Venn diagrams, was distributed to 120 pre-service teachers who had been learning a basic course in set theory, from two colleges of education. Our findings highlight the fact that the verbal representation was the easiest, in other words, the students succeeded more in switching from the verbal and the verbal-logical to the visual representation, more than in switching from the formal representation to the visual one. Moreover, the main sources of errors were: Non identification of the problem variable or subject; adding new condition enlarges the set; Partial visual attribution to Venn diagrams and the inclusion relation is not checked element-wise and there were some misconceptions that contributed to errors.

Key words: Set theory, inclusion relation, representation, visualization, Venn Diagram.

1. Introduction

One of the researchers of this study raised the following problem as an exercise in the course called "Set Theory", which is intended to a group of students who are specializing in mathematics in one of the colleges for education in Israel.

> *Problem: Set A represents all the people who read a daily newspaper every day. Set B represents all the people who read the same newspaper twice a week. Set C represents all the people who read the newspaper on Friday and Sunday. Describe using Venn Diagram the inclusion / non-inclusion relation between the sets A, B and C.*

The researcher raised the problem also in front of different groups at different levels during different semesters and years of study, who studied the course "Set Theory". Nearly all the students' answers to this problem were wrong.

This phenomenon aroused the researchers' curiosity, who became interested in investigating the problem thoroughly.

2. Analysis of the Problem

2.1 A detailed solution to the problem

If an element x belongs to set A, then x is a person who reads the newspaper daily, i.e., every day of the week, including Fridays and Sundays. Therefore, x is also an element of set C. Thus, according to the inclusion definition of sets, set A is included in set C. If x is an element of set C, then x is a person who reads the newspaper on Fridays and Sundays, and therefore x DOES read the newspaper twice a week, and thus, x belongs also to set B.

Corresponding author: Amal Sharif-Rasslan, Ph.D., research fields: advanced mathematical thinking, applied mathematics especially in cardiology. E-mail: amalras@macam.ac.il

2.2 An additional solution that clarifies the formal-logical side of the problem

Let A_i be the set of people who read the newspaper on day i, $(i=1,2,3,4,5,6,7)$. Thus, it is possible to describe each of the sets that appear in the problem by means of sets A_i :

$A=\bigcap_{i=1}^{7} A_i$, $C=A_1 \cap A_6$ - and $B=\bigcup_{\substack{i<j \\ 1\le i,j\le 7}} (A_i \cap A_j)$

We conclude that the answer to the given problem stems from the following properties:

1) Intersection between sets is included in each of the sets that create intersection;

2) The union of sets includes each of the sets that create the union.

Hence, $A \subseteq C \subseteq B$.

The analysis of the problem and its solution indicates that there are different issues lie in this problem, which are:

1) The Verbal Issue: the problem is expressed in words;

2) The Visual Issue: the problem includes visual descriptions (Venn Diagrams);

3) The Logical Issue: the problem includes explicit quantifiers and implicit connectives (as it was clarified in the Formal-logical solution).

In this study, we focus on connections between different representations of sets. We will mainly deal with:

1) The Verbal Representation, where the set is represented verbally, or in an explicit way. For example: let A be the set of students in some class. B is the set of boys and C is the set of girls in the same class;

2) The Verbal-Logical Representation, where the set is described verbally and includes quantifiers or connectives explicitly or implicitly. The main problem that was introduced at the beginning of this paper is of this type;

3) The Visual Representation, where the set is described by means of Venn Diagrams;

4) The Formal-Symbolic Representation, where the set is described by means of set-builder notation or symbols. For example:

$A=\{x \mid x \in N\}$, $B=\{x \mid x \in Z\}$, $C=\{x \mid x \in Q\}$

or

X, Y are any sets , $A=X \cup Y$, $B=X \cap Y$.

Mathematics education literature shows that many studies dealt with logical aspects; i.e. the four cards problem and problems that are derived from it [1-3]. These studies dealt mainly with the logical aspect ($p \to q$). Besides, they dealt with the arguments and justifications of the examinees.

However, as we explained in the analysis of the problem, we are discussing a problem which deals with various points of view that include the logical aspect involving quantifiers and connectives.

Other studies in mathematics education dealt with the basic concepts in set theory from different points of view [4-11]. Bagni [4] studied the pupils' difficulties in distinguishing between two basic concepts in set theory: the concept of "inclusion" and the concept of "belonging". In the same study, Bagni [4] reported two case studies of two students; one is 11 years old and the other is 15 years old. Bagni [4] checked the degree of understanding of the two students of the concepts of "inclusion" and "belonging" depending on Venn's diagram. Bagni [4] argued that the students' difficulties related to their failure to grasp the fact that the concept of inclusion is based in a synthetic view of sets: following Cantor, sets are comprised of individual objects and to claim that one set is included in another one requires an analysis of all the individual objects forming the first set [4, p. 274].

Zazkis and Gunn [7] reported about pre-service teachers' understanding of basic concepts in set theory: the concept of set, element of set, cardinality, subset, and empty set. It is worthwhile mentioning that in the study of Zazkis and Gunn [7], pre-service teachers participated in a computer-integrated project and used the ISETL computer language. The findings of the study indicated the complexity of the pre-service teachers' understanding of the concept of "element" in the "set" when the element itself is a "set". In addition, the study showed that there are difficulties in the description of the concept of "empty set".

Despite the passage of time, it is important to draw the attention to a document that was published in USA in 1969 by Neimark and Slotnick [9]. The researchers reported in that document the development of the connectives "and" and "or", and the quantifiers among the different populations in the elementary schools, junior high schools and students who study psychology at Douglass College in USA. In addition, they dealt with "intersection" and "union" of sets. The main finding in their report was that the answers of the "examinees" were correct to a large extent when the sets were described verbally and not with aid of pictures. However, the description by means of pictures was easier than the description with the aid of Venn's diagrams. Another finding that Neimark and Slotnick [9] mentioned in their document is that very young pupils internalized the concept of "inclusion" and "non-inclusion" of sets; pupils from grade 4 and above (including students) internalized the connectives in a good way. Intersection of sets was understood to all the examinees, but the union was understood only to students; for all the other examinees, the concept was not understood.

In mathematics it is regular to described sets by different representations: verbal representation, which lists all the elements of the set; representation by means of set-builder notation; representation by the aid of symbols such as N, Q, R, etc.; and visual representation by the aid of Venn's diagrams [12-13].

Dreyfus [14] argues that representation of concepts has a very important function in mathematics. One of the interpretations of "representations" is that it is a "mental representation". Different people attribute other representations of the same concept, while another interpretation of "representation" is that it is a "symbolic representation", as symbols require connections between signs and meaning. Therefore, in order to represent a certain concept, it is necessary to create an "imitation" to it [15-17].

Visualization is one process of creating mental representations [14]. According to Kaput's theory [18], the creation of "mental representation" depends on representation of systems. A person can create a single mental representation or a number of mental representations that compete with the same mathematical concept. Lakoff & Núñe [19] developed the theory of Metaphor of Containers (ibid. p. 45). In this theory, sets are conceived as "containers".

Switching from representation of a specific concept to another representation of the same concept is not easy because the structure is complex. Besides, the students mostly limit themselves to working with the same representation [14]. Dreyfus in his work [14] describes the process of "translation" of one representation to another as a passage from a specific way of wording of a concept, a mathematical argument or a problem to another representation.

3. Methodology

3.1 The Objectives of the Study

We focus in this study on the relation of inclusion/ non-inclusion between sets that are described by means of different representations. The main objective of the study is to examine the linkage between different representations of sets and the relations between them, and the source of errors during the passage between different representations of sets among pre-service teachers.

3.2 Questions of the Study

1) To what extent do pre-service teachers of mathematics connect between different representations of sets and the relations between sets?

2) What are the sources of errors made by pre-service teachers in connecting between different representations of sets and the inclusion relation between them?

3) Do pre-service teachers identify the variable of the problem (topic of the problem) in verbal and logical-verbal representations?

4) Do pre-service teachers understand and apply the property: “Adding new conditions reduces the set”.

3.3 Hypotheses of the Study

1) Pre-service teachers succeed more in solving problems on relations of inclusion that are introduced in verbal representation than in solving problems that are introduced in formal representation;

2) Pre-service teachers succeed more in solving problems on relations of inclusion that are introduced in formal representation than in solving problems that are introduced in verbal-logical representation.

3.4 Population of the Study

The sample of the study consists of 120 pre-service teachers who are specializing in mathematics at two colleges of education: (56 students from one college and 64 students from the other). All the students learned the course of “Set Theory”.

Distribution of the number of students according the year of study is detailed in Table 1.

Table 1 Distribution of pre-service teachers according to years of study.

Year of study	1	2	3	4
Number of students	29	34	40	17

3.5 Tools of the Study

In order to give answers to the questions of the study, we built a questionnaire that includes two kinds of questions: open questions, and multiple choice questions.

The Questionnaire consisted of 10 questions that dealt with different aspects of set description. In questions 1-9, the student is asked to express the inclusion/non-inclusion of sets given in one representation that we defined in the theoretical background (verbal, verbal-logical, or formal-symbolic) to the visual representation.

1) In order to identify the pre-service teachers’ errors, the questionnaire included multiple choice problems;

2) In order to discover the sources of errors, the pre-service teachers were asked to justify their choice;

3) Three questions aimed to check the correlation between the verbal-representation (i.e., explicit description of the element of the set), and the visual representation;

4) Three questions aimed to check the correlation between the verbal-logical representation and the visual representation;

5) Three questions aimed to check the correlation between the formal-symbolic representation and the visual representation;

6) The aim of question 10 was to check how pre-service teachers conceive of the concept of “inclusion” of two sets.

3.6 Validity and Reliability

The validity and reliability of the questionnaire was tested in two stages:

1) The questionnaire was given to two specialists (who are lecturers in mathematics at an academic college). Each specialist gave his notes on the quality of the questionnaires, and also answered the questions;

2) The questionnaire was passed to two groups of students. One group consisted of 10 students who were specializing in mathematics and were in the last semester of their studies. The second group included 10 students who studied the course “Set Theory”. The two groups of students were from two different colleges;

The questionnaire was updated after the pilot check of validity and reliability. After the update, the questionnaire was distributed among the 120 pre-service teachers who studied the course “Set Theory” at the two colleges.

3.7 Statistical Analysis

Statistical analyses were conducted using SPSS software version 17. Means and Standard Deviations, a pair t-Test and Pearson's Correlation Coefficient were performed, considering $p<0.01$ as statistically significant.

4. Results

In order to validate or negate the two hypotheses, the t-test was performed. The results that were received are detailed in the following table. Table 2, introduces means of the correct answers (out of 3 questions) for each representation, and Table 3 introduces the results of comparison according to a Pair t-Test.

Table 2 Means and standard deviations of correct answers.

Representation	Mean	N	Std. deviation	Std. error mean
Verbal	2.5417	120	0.56354	0.05144
Formal	1.3000	120	0.86578	0.07903
Verbal-logical	2.0250	120	0.95673	0.08734

According to Tables 2 and 3, the mean of the number of correct answers in the verbal representation is the highest. This confirms the researcher's first hypothesis: pre-service teachers succeed more in solving problems on relations of inclusion that are introduced in verbal representation (2.54 ± 0.564) than in solving problems that are introduced in formal representation (1.3 ± 0.866), $p<0.001$. Moreover, the number of the correct answers in the verbal representation is higher than the verbal-logical representation (2.03 ± 0.957), $p<0.001$. In other words, the pre-service teachers succeeded more in solving the problems that were introduced in the verbal representation than in solving the problems that were introduced in the verbal-logical representation.

On the other hand, according to the findings which are presented in the tables above, the pre-service teachers succeeded more in solving the problems that are introduced in the verbal-logical representation than in solving problems that are introduced in the formal representation, $p<0.001$, which is contrary to the second hypothesis of the research.

It is important to point out that one of the problems, which belongs to verbal-logical representation, includes two parts; the first of which paved the way to a correct solution: "write the elements of each of the sets" and the second was to draw the correct Venn diagram which describes the inclusion relation between the sets. When we compared the mean of the correct answers in the verbal-logical components, without the first part of this questions, (1.2417 ± 0.698), with the mean of the formal component (1.3 ± 0.866), we concluded that there is no difference between them, $p = 0.555$. This explains the importance of the first part in this question, i.e. adding a phrase like "write the elements of each of the sets" makes the question easier.

Table 4 introduces the percentages of the correct answers of all the questions, grouped according to different representations; column 2 in Table 4 introduces the classification of each representation in the questionnaire.

As a result of the information in Table 4, we had to check the correlation between "difficult questions" in the different components. A "difficult question" in each component is a question with the least correct answers. We focused specifically on the comparison of question 8, which was the most difficult one in the formal-symbolic component with question 7, which was the most difficult in the verbal component. Besides, we compared question 8 with every question in the verbal-logical component, because we noticed in the findings of the study that pre-service teachers succeeded more in solving problems that are introduced in the verbal-logic representation than in solving problems introduced in the formal representation, which is in contrary to the expectation we posed to ourselves in the second hypothesis of the research study. It was important to us to investigate this issue in a more profound way.

Table 3 Comparison between representations using Paired Samples t-Test.

	Paired differences			t	df	Sig.*
	Mean	Std. deviation	Std. error mean			
Verbal: Formal	1.24	0.92578	0.08451	14.69	119	0.000
Verbal: Verbal-logical	0.517	1.06891	0.09758	5.295	119	0.000
Formal: Verbal-logical	-0.725	1.23644	0.11287	-6.42	119	0.000

(*) 2-tailed.

Table 4 Distributions of correct answers (%) per representation.

Representation	Number of question	Correct answers (%)	Incorrect answers (%)
	1	99.167	0.833
Verbal	3	97	3
	7	58.33	41.67
	5	59.17	39.17
Formal-symbolic	4	48.33	51.67
	8	21.67	78.33
	2	55.83	44.17
Verbal-logical	6	78.33	21.67
	9	0.83	99.17

There is no correlation between question 8 and question 2, according to Cramer's correlation coefficient, $r_c = 0.103$. However, there is a limited weak border-line correlation between question 8 and question 9, $r_c = 0.174$, $p = 0.056$. Moreover, it was found that there is no correlation between question 8 and the first part of question 6. In contrast to this, we concluded that there is a weak correlation between question 8 and the second part of question 6, $r_c = 0.192$, $p = 0.035$. It is important to point out that there is a significant strong correlation between the two parts in question 6. 82% of those who answered correctly the first part answered correctly the second part, $r_c = 0.54$, $p < 0.001$.

In order to complete the answers of research questions (2.-4.) we had analyzed the students' explanations qualitatively. The main sources of errors were: Non identification of the problem variable or subject (third question); Adding new condition enlarges the set (fourth question); Partial visual attribution to Venn diagrams and the inclusion relation is not checked element-wise. Moreover, there were some misconceptions that contributed to errors, e.g., "If a set A contains a common element of some sets then A contains all the sets" and "Triangles are not polygons" or "each polygon is a triangle and there are triangles which are not polygons".

5. Discussion

The principles of set theory constitute a basic and fundamental part in the teaching program for specialization in mathematics at academic institutions for pre-service teacher in both the elementary and the secondary school tracks.

On the one hand, nearly all the courses of mathematics in the teaching program for specialization in mathematics in these tracks, such as: transformations, rational numbers, probability theory, introduction to the number theory, etc., include a chapter, which constitutes an introduction to set theory [5-7].

On the other hand, the concept of "set" and the basic concepts in the set theory is an important chapter [7, p. 135] to the understanding of addition and subtraction operations of natural numbers. In other words, the principles of set theory constitute a basis for the methods of teaching mathematics in general and arithmetic in particular.

This research assisted in spotting and understanding the difficulties that pre-service teacher encounter in learning basic concepts and different descriptions of the sets. It is worthwhile mentioning that the findings of the study may influence and repercussion on the methods of teaching the subject of mathematics in general and at elementary and high schools in particular.

Our research questionnaire focused on the inclusion relation between sets, by switching this relation from the verbal, formal and verbal-logical to the visual representation, i.e., to Venn diagrams. The verbal and verbal-logical representations concentrate on natural language; our findings highlight the fact that the verbal one was the easiest. It should be noted that the verbal and verbal-logical are based on natural language to a large extent. It was pointed by many researchers who believe that natural language plays significant role in students' efforts to learn mathematics [20-21], which explain the fact that the students succeeded more in switching from the verbal and the verbal-logical to the visual representation, more than in switching from the formal representation to the visual one.

This research was a trigger of the question that was mentioned at the beginning of the paper, which was question 9 in the questionnaire. The findings emphasized the source of the wrong answers of the pre-service teachers for this question; it was found that 119 of 120 students confused the variable of the question, in other words the question was about "the people who read the newspaper", while the students referred to "the days in the week the newspaper was read". This means that there is "non- identification of the problem variable or subject", which was concluded in our findings, and was repeated in more than one question.

Adding a note/part to a question such as "write the elements of each of the sets" may assist the student to solve the problem correctly; in other words adding such phrase may clarify the natural language.

6. Conclusions and Recommendations

Out of this study we may recommend the following:

1) The subject of different representations of sets must be emphasized together with switching from one representation to another;

2) We ought to take care of the mathematical language;

3) We have to consider the order of teaching mathematical matters and the connection between different concepts;

4) Formal definitions of mathematical concepts must be emphasized;

5) Basic principles and misconceptions should be raised in classroom discourse.

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