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# First Integral of Differential Systems of Rational Nonlinearities with a Star Node

Khalil I.T. Al-Dosary

*Department of Mathematics, University of Sharjah, Sharjah 27272, United Arab Emirates*

**Abstract:** In this paper we gave an expression for invariant algebraic curve of differential systems of rational nonlinearities with a star node of the form

$$\dot{x} = \lambda x + \frac{P_{n1}(x,y)}{P_{n2}(x,y)} \qquad \dot{y} = \lambda y + \frac{Q_{m1}(x,y)}{Q_{m2}(x,y)}$$

Where $\lambda \neq 0$ and $n_1, n_2, m_1$ ,and $m_2$ are integers and $P_i$, $Q_j$ , are homogeneous polynomials of degree *i, j* respectively such that $n_1 - n_2 = m_1 - m_2 \geq 2$. Then we proved that these systems are Darboux integrable and introduced an explicit expression of a Liouvillian first integral.

**Key words:** Rational system, integrability, star node, Darboux integrability, Liouvillian function, invariant algebraic curve.

## 1. Introduction

A two dimensional rational vector field defined on the real plane is a vector field of the form

$$X(x,y) = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y} \qquad (1)$$

where *P, Q* are rational functions i.e. division of two homogeneous polynomials, namely,

$$P(x,y) = \frac{P_{n1}(x,y)}{P_{n2}(x,y)} \quad Q(x,y) = \frac{Q_{m1}(x,y)}{Q_{m2}(x,y)} \qquad (2)$$

where $P_i(x,y)$ , $Q_j(x,y)$ are homogeneous polynomials of degree *i, j* respectively, and $P(x,y), Q(x,y)$ are coprime in the ring $R[x,y]$. Let *U* be an open subset of $R^2$ . If there exists a non-constant $C^1$ function $H: U \to R$ , which is constant on all the solutions of the differential system $(\dot{x},\dot{y}) = (P,Q)$ , contained in *U*, then we say that *H* is a first integral of the differential system on *U*, and that system is integrable on *U*; in other words $\frac{\partial H}{\partial x}P + \frac{\partial H}{\partial y}Q = 0$ on *U*. It is well known that for systems defined on the plane $R^2$, the existence of a first integral determines their phase portrait. Thus for differential systems a natural question arises for a given system, how to recognize if the system has a first integral? If $f \in R[x,y]$ , then $f(x,y) = 0$ is an algebraic curve. We say that $f(x,y) = 0$ is an invariant curve for the vector field *X* = (*P*, *Q*) if X*f* = K*f*, $f \in R[x,y]$ . In this case K is called the cofactor of *f*. Its degree is lower than the degree of *P*, *Q*. The expression which defines K is written as

$$\frac{\partial f}{\partial x}P + \frac{\partial f}{\partial y}Q = Kf \qquad (3)$$

Recall that a limit cycle of a system (1) is an isolated periodic solution in the set of all periodic solutions of the system. The problem of the integrability of the

**Corresponding author:** Khalil I.T. Al-Dosary, Ph.D., professor, research fields: scientific computing, mathematical analysis, control, optimization. E-mail: dosary@sharjah.ac.ae.

planar differential systems is very classical, but there is another classical problem related with the planar differential systems, the second part of the 16th Hilbert problem. This problem essentially consists in finding a uniform upper bound for the maximum number of limit cycles that a planar differential systems of a given degree can exhibit. For more details see for instance [1] and [2] and references therein. We can say roughly that an elementary function is a function which is composition of polynomials, exponential, logarithmic and algebraic functions. A Liouvillian function is a function that can be expressed by quadratures of elementary functions. A differential system is Darboux integrable if it has a first integral which is a Liouvillian function. For more details and precise definitions of elementary, Liouvillian functions, and Darboux integrability see for instance [3] and [4]. One can see from (3) that on points of the algebraic curve $f(x,y)=0$, the gradient of $f$ is orthogonal to the vector field $X = (P, Q)$. Hence at every point of $f = 0$ $X = (P, Q)$ the vector field $X$ is tangent to the curve $f = 0$, so the curve $f = 0$ is formed by trajectories of the vector field $X$. The terminology “invariant algebraic curves” is strongly related to the Darboux theory of integrability, see [5].

In [6] the authors proved that the polynomial differential systems of the form

$$\dot{x} = \lambda x + \mathrm{P_n}(\mathrm{x,y}) \qquad \dot{y} = \lambda y + Q_n(x,y) \qquad (4)$$

defined in $R^2$, $\lambda \neq 0$, and $\mathrm{P_n}(\mathrm{x,y})$, $Q_n(x,y)$ are homogeneous polynomials of degree $n$, are Darboux integrable and have no periodic solutions, and consequently no limit cycles. In [7] the authors provide an explicit Liouvillian first integral for them and an invariant algebraic curve.

In this paper, we extend these results about the integrability of the systems with $P_n$, $Q_n$ not necessarily polynomials but rational functions of the form 2. We give an expression for invariant algebraic curve for the systems. We show that these systems are Darboux integrable for any degree then we introduce an expression of a Liouvillian .rst integral.

## 2. Main Results

We consider differential systems of rational nonlinearities with a star node at the origin of the form

$$\begin{aligned} \dot{x} &= \lambda x + \frac{P_{n1}(x,y)}{P_{n2}(x,y)} \\ \dot{y} &= \lambda x + \frac{Q_{m1}(x,y)}{Q_{m2}(x,y)} \end{aligned} \qquad (5)$$

defined in $R^2$ where $\lambda \neq 0$, $n_1 - n_2 = m_1 - m_2 (= n)$ is an integer greater than one and $P_i$, $Q_j$, are homogeneous polynomials of degree $i$, $j$ respectively and coprime in the ring $R[x,y]$. The integer $n$ refers to the degree of the rational system. Hereafter in this paper the systems are assumed to be differentiable.

The following results show that the system (5) for all integer $n \succ 1$ always are Darboux integrable and provide an explicit Liouvillian first integral for them and an invariant algebraic curve. Before introducing the Theorem, we give the following Lemma which is a generalization for the Euler’s formula for homogeneous polynomials in $x, y$ of degree $n$.

**Lemma 1.**

For any integers $n_1, n_2$ and any two homogeneous polynomials $P_{n1}$, $P_{n2}$ in $(x,y)$ of degrees $n_1$ and $n_2$ respectively,

$$x\frac{\partial}{\partial x}\left(\frac{P_{n1}}{P_{n2}}\right) + y\frac{\partial}{\partial y}\left(\frac{P_{n1}}{P_{n2}}\right) = (n_1 - n_2)\left(\frac{P_{n1}}{P_{n2}}\right) \qquad (6)$$

**Proof.** Direct computation.

Recall that if $n_2 = 0$, we get the usual Euler's formula for homogeneous polynomials Pn1 in two variables $x, y$ of degree $n_1$.

**Theorem 1.** The curve

$$x\frac{Q_{m1}(x,y)}{Q_{m2}(x,y)} - y\frac{P_{n1}(x,y)}{P_{n2}(x,y)} = 0$$

is an invariant algebraic curve of system (5).

**Proof.** Hereafter we shall use the following notations

$$F(x,y) = x\frac{Q_{m1}(x,y)}{Q_{m2}(x,y)} - y\frac{P_{n1}(x,y)}{P_{n2}(x,y)},$$
$$P(x,y) = \lambda x + \frac{P_{n1}(x,y)}{P_{n2}(x,y)}, \qquad (7)$$
$$Q(x,y) = \lambda y\frac{Q_{m1}(x,y)}{Q_{m2}(x,y)}.$$

Here we have

$$\frac{\partial F}{\partial x}P + \frac{\partial F}{\partial y}Q = \frac{\partial F}{\partial x}\lambda x + \frac{\partial F}{\partial y}\lambda y + \frac{\partial F}{\partial x}\frac{P_{n1}}{P_{n2}} + \frac{\partial F}{\partial y}\frac{Q_{m1}}{Q_{m2}} \qquad (8)$$

Using (7), the first two terms of the right-hand side of (8) are

$$\frac{\partial F}{\partial x}\lambda x + \frac{\partial F}{\partial y}\lambda y = \left(\frac{Q_{m1}}{Q_{m2}} + x\frac{\partial}{\partial x}\frac{Q_{m1}}{Q_{m2}} - y\frac{\partial}{\partial x}\frac{P_{n1}}{P_{n2}}\right)\lambda x + \left(x\frac{\partial}{\partial y}\frac{Q_{m1}}{Q_{m2}} - \frac{P_{n1}}{P_{n2}} - y\frac{\partial}{\partial y}\frac{P_{n1}}{P_{n2}}\right)\lambda y$$
$$= \lambda\left[\begin{array}{l} x\left(x\frac{\partial}{\partial x}\frac{Q_{m1}}{Q_{m2}} + y\frac{\partial}{\partial y}\frac{Q_{m1}}{Q_{m2}} + \frac{Q_{m1}}{Q_{m2}}\right) \\ -y\left(\frac{P_{n1}}{P_{n2}} + x\frac{\partial}{\partial x}\frac{P_{n1}}{P_{n2}} + y\frac{\partial}{\partial y}\frac{P_{n1}}{P_{n2}}\right)\end{array}\right]$$

Applying Lemma 1, we obtain

$$\frac{\partial F}{\partial x}\lambda x + \frac{\partial F}{\partial y}\lambda y = \lambda\left[\begin{array}{l} x\left((m_1 - m_2)\frac{Q_{m1}}{Q_{m2}} + \frac{Q_{m1}}{Q_{m2}}\right) \\ -y\left((n_1 - n_2)\frac{P_{n1}}{P_{n2}} + \frac{P_{n1}}{P_{n2}}\right)\end{array}\right]$$
$$= \lambda(n_1 - n_2 + 1)\left[x\frac{Q_{m1}}{Q_{m2}} - y\frac{P_{n1}}{P_{n2}}\right]$$
$$\sin ce\ n_1 - n_2 = m_1 - m_2$$
$$= \lambda(n_1 - n_2 + 1)F(x,y).$$

The third and fourth terms of the right-hand of (8) are

$$\frac{\partial F}{\partial x}\frac{P_{n1}}{P_{n2}} + \frac{\partial F}{\partial y}\frac{Q_{m1}}{Q_{m2}} = \left(\frac{Q_{m1}}{Q_{m2}} + x\frac{\partial}{\partial x}\frac{Q_{m1}}{Q_{m2}} - y\frac{\partial}{\partial x}\frac{P_{n1}}{P_{n2}}\right)\frac{P_{n1}}{P_{n2}} \qquad (9)$$
$$+\left(x\frac{\partial}{\partial y}\frac{Q_{m1}}{Q_{m2}} - \frac{P_{n1}}{P_{n2}} - y\frac{\partial}{\partial y}\frac{P_{n1}}{P_{n2}}\right)\frac{Q_{m1}}{Q_{m2}} \qquad (10)$$

Applying Lemma 1, we get

$$x\frac{\partial}{\partial x}\frac{Q_{m1}}{Q_{m2}} = (m_1 - m_2)\frac{Q_{m1}}{Q_{m2}} - y\frac{\partial}{\partial y}\frac{Q_{m1}}{Q_{m2}}$$
$$and \quad y\frac{\partial}{\partial y}\frac{P_{n1}}{P_{n2}} = (n_1 - n_2)\frac{P_{n1}}{P_{n2}} - x\frac{\partial}{\partial x}\frac{P_{n1}}{P_{n2}}$$
$$and \quad m_1 - m_2 = n_1 - n_2$$

Therefore

$$\frac{\partial F}{\partial x}\frac{P_{n1}}{P_{n2}}+\frac{\partial F}{\partial y}\frac{Q_{m1}}{Q_{m2}}$$
$$=\left(\frac{Q_{m1}}{Q_{m2}}+(n_1-n_2)\frac{Q_{m1}}{Q_{m2}}-y\frac{\partial}{\partial y}\frac{Q_{m1}}{Q_{m2}}-y\frac{\partial}{\partial x}\frac{P_{n1}}{P_{n2}}\right)\frac{P_{n1}}{P_{n2}}$$
$$+\left(x\frac{\partial}{\partial y}\frac{Q_{m1}}{Q_{m2}}-\frac{P_{n1}}{P_{n2}}-(n_1-n_2)\frac{P_{n1}}{P_{n2}}+x\frac{\partial}{\partial x}\frac{P_{n1}}{P_{n2}}\right)\frac{Q_{m1}}{Q_{m2}}$$
$$=-y\left(\frac{\partial}{\partial y}\frac{Q_{m1}}{Q_{m2}}\right)\frac{P_{n1}}{P_{n2}}-y\left(\frac{\partial}{\partial x}\frac{P_{n1}}{P_{n2}}\right)\frac{P_{n1}}{P_{n2}}$$
$$+x\left(\frac{\partial}{\partial y}\frac{Q_{m1}}{Q_{m2}}\right)\frac{Q_{m1}}{Q_{m2}}+x\left(\frac{\partial}{\partial x}\frac{P_{n1}}{P_{n2}}\right)\frac{Q_{m1}}{Q_{m2}}$$
$$=\left(\frac{\partial}{\partial x}\frac{P_{n1}}{P_{n2}}+\frac{\partial}{\partial y}\frac{Q_{m1}}{Q_{m2}}\right)\left(x\frac{Q_{m1}}{Q_{m2}}-y\frac{P_{n1}}{P_{n2}}\right)$$
$$=\left(\frac{\partial}{\partial x}\frac{P_{n1}}{P_{n2}}+\frac{\partial}{\partial y}\frac{Q_{m1}}{Q_{m2}}\right)F(x,y)$$

Therefore equation (8) will be

$$\frac{\partial F}{\partial x}P+\frac{\partial F}{\partial y}Q$$
$$=\lambda(n_1-n_2+1)F(x,y)+\left(\frac{\partial}{\partial x}\frac{P_{n1}}{P_{n2}}+\frac{\partial}{\partial y}\frac{Q_{m1}}{Q_{m2}}\right)F(x,y)$$
$$=\left(\lambda(n_1-n_2+1)+\frac{\partial}{\partial x}\frac{P_{n1}}{P_{n2}}+\frac{\partial}{\partial y}\frac{Q_{m1}}{Q_{m2}}\right)F(x,y)$$

Hence $F(x,y)=0$ is an invariant algebraic curve of the rational differential system (5). This completes the proof of the Theorem.

**Theorem 2.**

System (5) is Darboux integrable for any integer $n_1$, $n_2$, $m_1$, and $m_2$ such that $m_1-m_2=n_1-n_2\geq 2$ and $\lambda\neq 0$ with the Liouvillian first integral

$$H(x,y)=$$
$$(x^2+y^2)^{\frac{n_1-n_2-1}{2}}\exp\left((1-n_1+n_2)\int^{\arctan\frac{y}{x}}\frac{f(\theta)}{g(\theta)}d\theta\right)$$
$$+$$
$$(1-n_1+n_2)\lambda\int^{\arctan\frac{y}{x}}\frac{\exp\left((1-n_1+n_2)\int^{\theta}\frac{f(\mu)}{g(\mu)}d\mu\right)}{g(\theta)}d\theta$$

*where*

$$\begin{aligned}F(\theta)&=\cos\theta\frac{P_{n1}(\cos\theta,\sin\theta)}{P_{n2}(\cos\theta,\sin\theta)}\\&\quad+\sin\theta\frac{Q_{m1}(\cos\theta,\sin\theta)}{Q_{m2}(\cos\theta,\sin\theta)}\\G(\theta)&=\cos\theta\frac{Q_{m1}(\cos\theta,\sin\theta)}{Q_{m2}(\cos\theta,\sin\theta)}\\&\quad-\sin\theta\frac{P_{n1}(\cos\theta,\sin\theta)}{P_{n2}(\cos\theta,\sin\theta)}\end{aligned}\qquad(11)$$

**Proof.**

In polar coordinates $(\gamma,\theta),\gamma=\cos\theta,\gamma=\sin\theta,$ system (5) becomes

$$\dot{r}=\lambda r+f(\theta)r^{(n_1-n_2)},\ and\ \dot{\theta}=g(\theta)r^{(n_1-n_2-1)}\qquad(12)$$

where $f(\theta)$ *and* $g(\theta)$ are the functions de.ned in (11) in the statement of the Theorem. The differential system (12) for $g(\theta)\neq 0$ can be written in an equivalent differential system of the form

$$\frac{dr}{d\theta}=\frac{\lambda}{g(\theta)}\frac{1}{r^{(n_1-n_2-1)}}+\frac{f(\theta)}{g(\theta)}r\qquad(13)$$

This is a Bernoulli differential equation. If we change the variable $r$ by the new variable $\rho=r^{(n_1-n_2-1)}$, then the equation (13) will be transformed to the following linear differential equation

$$\frac{d\rho}{d\theta} = (n_1 - n_2 - 1)\frac{\lambda}{g(\theta)} + (n_1 - n_2 - 1)\frac{f(\theta)}{g(\theta)}\rho$$

Solving this first order linear differential equation and making use of the inverse of the used transformation, we find the first integral in the required form in the Theorem which is a Liouvillian function form. Therefore system (5) is Darboux integrable. Hence the proof is complete.

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