Journal of Mathematical Sciences 2 (2015) 17-21 **Betty Jones & Sisters Publishing**

The Simplex Method and 0-1 Polytopes

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Abstract: We will derive two results of the primal simplex method by constructing simple LP instances on 0-1 polytopes. First, for any 0-1 polytope and any of its two vertices, we can construct an LP instance for which the simplex method finds a path between them, whose length is at most the dimension of the polytope. This proves a well-known result that the diameter of any 0-1 polytope is bounded by its dimension. Next we show that the upper bound for the number of distinct solutions generated by the simplex method is tight. We prove these results by using a basic property of the simplex method.

Keywords: Linear programming, the number of solutions, the simplex method, 0-1 polytope.

1. Introduction

The simplex method developed by Dantzig [1] could efficiently solve a real world linear programming problem (LP), but good upper bound for the number of iterations has not been known yet. Kitahara and Mizuno [2] extended Ye's result [3] for the Markov decision problem and obtained an upper bound for the number of distinct solutions generated by the simplex method with Dantzig's rule of pivoting for the standard form *LP*

$$\begin{array}{lll} (P_0) & \min & c^T x, \\ \textit{subject to} & & Ax = b, x \geq 0 \end{array}$$

where A is an $m\times n$ matrix, $b \in R^m$, $c \in R^n$, and $x \in R^n$. The upper bound in [2] is expressed as

$$nm \frac{\gamma_P}{\delta_P} \log \left(m \frac{\gamma_P}{\delta_P} \right),$$

where δ_P and γ_P are respectively the minimum and the maximum values of all the positive elements of basic feasible solutions of (P_0). Recently, Kitahara and Mizuno [4] obtained a new upper bound

$$m \frac{\gamma_P \gamma'_D}{\delta_P \delta'_D} \quad (1)$$

for the number of distinct solutions generated by the primal simplex method with any rule, which chooses an entering variable whose reduced cost is negative at each iteration. In (1), δ'_D and γ'_D are respectively the minimum and the maximum absolute values of all the negative elements of dual basic solutions for primal feasible bases. These results are derived from the basic property of the primal simplex method that the objective function value strictly decreases whenever a basic solution is updated, even if the problem is degenerate. Note that the number of distinct solutions is not always identical to the number of iterations when the problem is degenerate.

The results of Ye's and ours gave us new insights to the simplex method. In this paper, we further study properties of the simplex method, especially when it is applied to LPs on 0-1 polytopes. By using the basic property of the primal simplex method, we will derive two results:

(R1) For any 0-1 polytope and any of its two vertices, we can construct an LP instance for which the simplex method finds a path between them, whose length is at most the dimension of the polytope;

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(R2)The upper bound (1) for the number of distinct solutions generated by the simplex method is tight.

Although we can easily prove these results by constructing simple LP instances on 0-1 polytopes, they are new as far as we know.

Let d be a positive integer and $S_1 \subset R^d$ be any 0-1 polytope, which is a convex hull of 0-1 vectors in R^d. The following well-known result is proved by Naddef [5].

(F1) there exists a path with the length of at most d between any two vertices on S_1.

This result is followed by (R1). Naddef showed the existence of the path. In this paper, we will show that the path is actually obtained by using the simlex method. (F1) implies that the diameter of any 0-1 polytope in R^d is bounded by the dimension d. Naddef [5] proved from (F1) that the Hirsch conjecture is true for 0-1 polytopes, although Santos [6] recently constructed a counterexample of the conjecture for general polytopes. Kleinschmit and Onn [7] extended (F1) to any $\{0,1,\cdots,k\}$-polytope in R^d and obtained that

(F2) there exists a path of the length of at most kd between any two vertices on the polytope.

It is also shown in Ziegler [8] that

(F3) for any linear function there exists a monotone path with the length of at most d between any given vertex and an optimal vertex on S_1,

where a path is said to be monotone when values of the linear function are monotone along the path. (F3) states existence of a short monotone path, but good upper bound for the length of a long monotone path between two vertices on S_1 has not been obtained so far. We do not know how many solutions are generated by the simplex method for LP on a 0-1 polytope. We will derive a simple upper bound for the number of solutions in Theorem 2, when coefficients of an objective function are integers. By using Theorem 2, the result (R1) is proved in Theorem 3.

We prove (R2) in Theorem 4 by constructing a simple LP instance on a 0-1 polytope, for which the simplex method generates exactly $m\frac{\gamma_P \gamma'_D}{\delta_p \delta'_D}$ distinct solutions. Knowing the tightness of an upper bound is important. It is shown in [4] that $m\frac{\gamma_P \gamma'_D}{\delta_p \delta'_D}$ is an upper bound for the number of distinct solutions generated by the simplex method. Our second result (R2) shows that the upper bound is tight. Hence it is impossible to get a better upper bound than $m\frac{\gamma_P \gamma'_D}{\delta_p \delta'_D}$, for example, $\frac{1}{2}m\frac{\gamma_P \gamma'_D}{\delta_p \delta'_D}$ or $\frac{\gamma_P \gamma'_D}{\delta_p \delta'_D}$.

2. The Simplex Method for an LP on a 0-1 Polytope

In this section, we consider the following *LP*

$$
\begin{array}{lll}
(P_1) & \min & c^T x, \\
 & \textit{subject to} & x \in S,
\end{array}
$$

where $S \subset R^d$ is a polytope, $c \in R^d$ is a constant vector, and $x \in R^d$ is a vector of variables. Let x^0 be a vertex of S. To solve (P_1), we use the primal simplex method from the initial vertex x^0 with any rule, which chooses an entering variable whose reduced cost is negative at each iteration. The simplex method actually solves a standard form *LP*, which is equivalent to (P_1), and generates a sequence of its basic feasible solutions. In this section, we identify a vertex of S with a basic feasible solution of the standard form *LP*. The next lemma states a basic property of the simplex method.

Lemma 1.

The objective function value decreases whenever a basic feasible solution is updated by the primal simplex method with any rule, which chooses an entering variable whose reduced cost is negative.

Note that when the problem is degenerate, a basic feasible solution may not be updated even if a basis is updated by the simplex method.

Let $M(P_1)$ be the maximum difference of objective function values between two vertices of S and $L(P_1)$ be the minimum positive difference of objective function values between two adjacent vertices of S. Two vertices are said to be adjacent when the segment connecting them is an edge of S. Since the objective function value decreases by at least $L(P_1)$ whenever a vertex is updated, the number of distinct vertices generated by the primal simplex method for solving (P_1) is bounded by

$$\frac{M(P_1)}{L(P_1)}. \qquad (2)$$

The idea of the discussion above is not new (see [7] for example) and the next theorem is easily derived from (2), but the result is not shown before in the literature as far as we know.

Theorem 2.

Let $S_1 \subset R^d$ be any 0-1 polytope and let c be any d-dimensional integral vector. Then the primal simplex method for solving the *LP*

$$(P_2) \quad \min \quad c^T x, \qquad \textit{subject to} \quad x \in S_1,$$

starting from any vertex of S_1 generates at most C distinct vertices, where $C = \sum_{i=1}^{d} |c_i|$.

Proof.

Since any vertex of S_1 is a 0-1 vector, the difference of objective function values between two vertices of S_1 is bounded by $C = \sum_{i=1}^{d} |c_i|$ that is, $M(P_2) \le C$.

The objective function value at any vertex is an integer, because the vector c is integral. So the minimum positive difference of objective function values between two adjacent vertices is at least one, that is,

$$L(P_2) \ge 1.$$

Hence the number of distinct vertices generated by the primal simplex method starting from any initial vertex is bounded by C from (2). □

A finite sequence $\{x^k \mid k = 0, 1, 2, \dots, \ell\}$ of vertices of S_1 is called a path of length ℓ between x^0 and x^ℓ on S_1, if any two consecutive vertices x^k and x^{k+1} are adjacent. A sequence of distinct vertices generated by the simplex method is a path. We obtain the next result by using Theorem 2 to a special *LP* instance.

Theorem 3.

Let $S_1 \subset R^d$ be any 0-1 polytope and x^s and x^t be any two vertices of S_1. Then we can construct an *LP* instance, for which the simplex method with Bland's rule [9] (or any anticycling rule) finds a path with the length of at most d between x^s and x^t.

Proof.

Let $x^t = (x_1^t, x_2^t, \dots, x_d^t)^T$. We define a vector $c = (c_1, c_2, \dots, c_d)^T$ by

$$c_i = \begin{cases} -1 & \textit{if} \quad x_i^t = 1, \\ 1 & \textit{if} \quad x_i^t = 0. \end{cases}$$

Obviously x^t is the unique optimal vertex of (P_2). The simplex method starting from x^s with Bland's rule always finds the optimal vertex x^t in a finite number of iterations. Then the number of distinct vertices generated by the simplex method is at most

$$C=\sum_{i=1}^{d}|c_i|=d$$

from Theorem 2. Hence the simplex method finds a path with the length of at most d. □

3. Tightness of the Upper Bound

We show that the upper bound (1) is tight in the following theorem.

Theorem 4.
The upper bound (1) is tight in the sense that there exists an LP instance for which the primal simplex method generates exactly $m\frac{\gamma_P\gamma'_D}{\delta_p\delta'_D}$ distinct solutions.

Proof.
We construct an LP instance on the m-dimensional cube

$$\begin{aligned} &\min && -e^T x,\\ &\textit{subject to} && x \le e,\\ &&& x \ge 0 \end{aligned}$$

or its standard form LP

$$(P_3)\quad \begin{aligned} &\min && -e^T x,\\ &\textit{subject to} && x+u=e,\\ &&& x\ge 0, u\ge 0, \end{aligned}$$

where $x=(x_1,x_2,...,x_m)^T$ is a vector of variables, $u=(u_1,u_2,...,u_m)^T$ is a vector of slacks, and $e=(1,1,...,1)^T$. The dual problem of (P_3) is

$$(D_3)\quad \begin{aligned} &\max && e^T y,\\ &\textit{subject to} && y\le e,\\ &&& y\le 0, \end{aligned}$$

where $y=(y_1,y_2,...,y_m)^T$ is a vector of dual variables.

It is easy to see that for any subset $K\subset\{1,2,\ldots,m\}$ the point $\left(x^k,u^k\right)$ defined by

$$\begin{aligned} &x_i^K=1, u_i^K=0 \ \textit{for any } i\in K,\\ &x_i^K=0, u_i^K=1 \ \textit{for any } i\notin K,\\ &x^K=\left(x_1^K,x_2^K,...,x_m^K\right)^T,\\ &u^K=\left(u_1^K,u_2^K,...,u_m^K\right)^T. \end{aligned} \tag{3}$$

is a basic feasible solution of (P_3) and conversely any basic feasible solution is expressed as (3) for some $K\subset\{1,2,\ldots,m\}$. Thus the feasible region of the problem (P_3) is a 0-1 polytope and we have that

$$\delta_p=1 \textit{ and } \gamma_p=1$$

where δ_P and γ_P are respectively the minimum and maximum values of all the positive elements of basic feasible solutions of (P_3). Similarly for any subset $K\subset\{1,2,\ldots,m\}$ the point y^K defined by

$$\begin{aligned} &y_i^K=-1 \textit{ for any } i\in K,\\ &y_i^K=0 \ \textit{ for any } i\notin K,\\ &y^K=\left(y_1^K,y_2^K,...,y_m^K\right)^T. \end{aligned}$$

is a basic solution of (D_3) and any basic solution is expressed as above. Hence

$$\delta'_D=1 \text{ and } \gamma'_D=1,$$

where δ'_D and γ'_D are respectively the minimum and the maximum absolute values of all the negative elements of basic solutions of (D_3) for primal feasible bases.

Note that the dual basic solution y^K is feasible only when $K=\{1,2,\ldots,n\}$. So the optimal solution of (P_3) is

$$x^* = (1, 1, ..., 1)^T, u^* = (0, 0, ..., 0)^T.$$

Let the initial solution be

$$x^0 = (0, 0, ..., 0)^T, u^0 = (1, 1, ..., 1)^T.$$

Since the feasible region of (P_3) is the m-dimensional cube, the length of the shortest path between (x^0,u^0) and (x^*,u^*) is m. So the primal simplex method starting from the initial solution (x^0,u^0) finds the optimal solution (x^*,u^*) by generating at least m distinct solutions. On the other hand, the number of distinct solutions generated is at most $m\frac{\gamma_P \gamma'_D}{\delta_P \delta'_D}$, which is equal to m.

Hence the primal simplex method generates exactly $m\frac{\gamma_P \gamma'_D}{\delta_P \delta'_D}$ distinct solutions. □

Acknowledgements

This research is supported in part by Grant-in-Aid for Young Scientists (B) 23710164 and Grant-in-Aid for Science Research (A) 20241038 of Japan Society for the Promotion of Science.

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