Journal of Mathematical Sciences 2 (2015) 66-73 **Betty Jones & Sisters Publishing**

Comparison of Stochastic Mortality Models: Application to Turkish Mortality Data

Funda KUL, Meral SUCU

Actuarial Science Department, Hacettepe University, Ankara 06800, Turkey

Abstract: With the rapid aging of the population, mortality modelling and forecasting has become growingly important.Many stochastic mortality models were constructed and used in the literature. In this work, applying the eight stochastic mortality models, Lee & Carter model [1] together with its seven well-known modifications and extensions. We check out the goodness of fit for available Turkish mortality dataspanning 1980-2012 based on BIC ranking criterion, unexplained variance and MAPE criteria. The autoregressive moving average (ARIMA) models are used to forecast the general index for the time and cohort period that goes from 2013 to 2030 in order to project life expectancy at birth.

Keywords: Modelling, mortality, stochastic, Turkey.

1. Introduction

With the rapid aging of the population, mortality modelling and projection has become growingly important, particularly work some institutions problems:

(1) increasing the expenses of the social security systems for goverments;

(2) predicting future cash-flows valuation, determining the capital requirements relating to the liabilities and remaining sustainable for annuity providers and pension plans.

In the fields of Actuarial Science and Demography, there have been attempt to find a favorable model that fits past mortality data [2]. Mortality modelling and projection techniques are seperated by *deterministic approach* and *stochastic approach*.

Mortality rate is defined as a function of age in deterministic approach [3-5]. In this approach, mortality is not change according to time. When mortality is modelled by this approach, significant deviations were observed in mortality projections [6]. Therefore, stochastic approach has been developed.

The well-known stochastic approach model is Lee & Carter [1]. This method has been widely used for modeling and projection mortality in literature. One reason for its popularity is the simplicity of the model and the straight forward way of projection the mortality ones the model is fitted. Despite of these advantages, Lee & Carter [1] model has several drawbacks. Many papers are written since Lee & Carter [1] have tried to eliminate this disadvantages by changing parameter estimation technique, adding more age-time interaction effect component, adding some age-time function components or adding cohort effect component [7].

In this paper, Lee & Carter [1] model and some its well-known modifications and extensions are fitted Turkish mortality data seperated by gender.

In generally, mortality data is two-dimensional. It is scheduled that the Lexis diagram is divided into unit square cells of size one year, by single year of age and calendar year [8]. In Lexis diagram, mortality rate is

Corresponding author: Funda KUL., Dr., prof. research fields: mortality modeling, claim modeling. E-mail: fundakaraman@hacettepe.edu.tr.

constant and just changes between cells. There are three effects defined in mortality data:age effect, time effect and cohort (birth time) effect [9]. Central mortality rate is defined by

$$m_{x,t} = \frac{D_{x,t}}{E_{x,t}} \quad (1.1)$$

where

$D_{x,t}$: shows that deaths during calendar year t aged x last birthday;

$E_{x,t}$: shows that exposure to death during calendar year t aged x last birthday.

The main assumption for the mortality modeling and projection is

$$m_{x+\xi,t+\tau} = m_{x,t}, \quad 0 \le \xi, \ \tau < 1 \quad (1.2)$$

Using the central mortality rate, annual death rate is defined by

$$q_{x,t} = 1 - \exp\left(-m_{x,t}\right) \quad (1.3)$$

The stochastic mortality models and their constraints are presented in Section 2. The Turkish mortality data description and parameter estimation method is given in Section 3. Results are given in Section 4, discussion of the results and a summary are given in Section 5.

2. Stochastic Mortality Models

2.1. Calibrating Stochastic Mortality Models

The well-known stochastic mortality models and parameter constraints are given in the Table 1, Table 2 and Table 3.

Table 1 Model formulation.

Model	Formula
M1	$\log\left(m_{x,t}\right)=\beta_x^{(1)}+\beta_x^{(2)}\kappa_t^{(2)}+\varepsilon_{x,t}$
M2	$\log\left(m_{x,t}\right)=\beta_x^{(1)}+\beta_x^{(2)}\kappa_t^{(2)}+\beta_x^{(3)}\kappa_t^{(3)}+\varepsilon_{x,t}$
M3	$\log\left(m_{x,t}\right)=\beta_x^{(1)}+\kappa_t^{(1)}+\gamma_{t-x}^{(3)}+\varepsilon_{x,t}$
M4	$\log\left(m_{x,t}\right)=\beta_x^{(1)}+\beta_x^{(2)}\kappa_t^{(2)}+\beta_x^{(3)}\gamma_{t-x}^{(3)}+\varepsilon_{x,t}$
M5	$\operatorname{logit}\left(q_{x,t}\right)=\kappa_t^{(1)}+\left(x-\bar{x}\right)\kappa_t^{(2)}+\varepsilon_{x,t}$
M6	$\operatorname{logit}\left(q_{x,t}\right)=\kappa_t^{(1)}+\left(x-\bar{x}\right)\kappa_t^{(2)}+\gamma_{t-x}^{(3)}+\varepsilon_{x,t}$
M7	$\operatorname{logit}\left(q_{x,t}\right)=\kappa_t^{(1)}+\left(x-\bar{x}\right)\kappa_t^{(2)}+c_x\kappa_t^{(3)}+\gamma_{t-x}^{(4)}+\varepsilon_{x,t}$
M8	$\log\left(m_{x,t}\right)=\beta_x^{(1)}+\kappa_t^{(1)}+\left(x-\bar{x}\right)\kappa_t^{(2)}+\left(x-\bar{x}\right)^{+}\kappa_t^{(3)}+\gamma_{t-x}^{(4)}+\varepsilon_{x,t}$

In Table 1, some functions are defined as

$$\bar{x} = \frac{1}{k}\sum_{i=x}^{x_k} i \quad (2.1)$$

$$\left(x-\bar{x}\right)^{+} = \max\left(\left(x-\bar{x}\right),0\right) \quad (2.2)$$

$$c_x = \left(x-\bar{x}\right)^2 - \frac{1}{k}\sum_{i=x}^{x_k}\left(i-\bar{x}\right)^2 \quad (2.3)$$

Table 2 Parameter constraints.

Model	
M1	$\sum_x \beta_x^{(2)}=1$, $\sum_t \kappa_t^{(2)}=0$
M2	$\sum_x \beta_x^{(2)}=1$, $\sum_t \kappa_t^{(2)}=0$, $\sum_x \beta_x^{(3)}=1$, $\sum_t \kappa_t^{(3)}=0$
M3	$\sum_t \kappa_t^{(1)}=0$, $\sum_{x,t} \gamma_{t-x}^{(3)}=0$
M4	$\sum_x \beta_x^{(2)}=\sum_x \beta_x^{(3)}=1$, $\sum_t \kappa_t^{(2)}=0$, $\sum_{x,t} \gamma_{t-x}^{(3)}=0$
M5	$\sum_t \kappa_t^{(1)}=0$, $\sum_t \kappa_t^{(2)}=0$
M6	$\sum_t \kappa_t^{(1)}=0$, $\sum_t \kappa_t^{(2)}=0$, $\sum_{x,t} \gamma_{t-x}^{(3)}=0$
M7	$\sum_t \kappa_t^{(1)}=0$, $\sum_t \kappa_t^{(2)}=0$, $\sum_t \kappa_t^{(3)}=0$, $\sum_{x,t} \gamma_{t-x}^{(4)}=0$
M8	$\sum_x \beta_x^{(1)}=1$, $\sum_t \kappa_t^{(1)}=0$, $\sum_t \kappa_t^{(2)}=0$, $\sum_t \kappa_t^{(3)}=0$, $\sum_{x,t} \gamma_{t-x}^{(4)}=0$

Table 3 Models used in this paper.

Model	Original Paper
M1	Lee & Carter [1]
M2	Renshaw & Haberman [10]
M3	Currie[11]
M4	Renshaw & Haberman [12]
M5	Cairns et al. [13]
M6	Cairns et al. [13]
M7	Cairns et al. [14]
M8	Plat [15]

2.2. Parameter Estimation

Models are fitted by maximum likelihood estimation, using the Poisson model for death counts. Because of the right hand side of the equations are non-linear in parameters. Log-likelihood function, that is:

$$l(\phi;D,E) = \sum_{t,x} D_{x,t} \log\left[E_{x,t}\ m_{x,t}(\phi)\right] - E_{x,t}\ m_{x,t}(\phi) - \log\left[D_{x,t}!\right] \quad (2.4)$$

where ϕ shows that unknown parameter class [2].

Model for the period effect we use a multivariate random walk with drift, that is,

$$\kappa_t^{(i)}=\kappa_{t-1}^{(i)}+\mu_k^{(i)}+\sigma_k^{(i)} Z_k^{(i)}(t) \quad (2.5)$$

where $\mu_k^{(i)}$ are the drifts, $\sigma_k^{(i)}$ are the volatilities and $Z_k^{(i)}(t)$ are the standard normal innovations correlated across the component independent of time [16] .For the cohort effect we use ARIMA (1,1,0), that is,

$$\gamma_c=\gamma_{c-1}+\mu_\gamma+\alpha_\gamma\left(\gamma_{c-1}-\gamma_{c-2}-\mu_\gamma\right)+\phi\ Z_\gamma(c) \quad (2.6)$$

where $Z_\gamma(c)$ are independent and identically distributed standard normal innovations, that are independent of the period effect innovations [16]. Parameters are estimated by using Lifemetrics R-code software.

3. Application

The models were fitted to age-specific rates of mortality for Turkish population between the years of 1980 to 2012. The data which included numbers of deaths and number of lives exposed to death were grouped into 5 year age bands for the purpose of fitting mortality rates. Figure 1 shows the female and male mortality rates and Figure 2 shows coefficient of variation of mortality rates.

Mortality rates are very volatile and vary from year to year and from age group to age group as displayed by Figure 1 and Figure 2. Also, it is seen that female mortality rates are more variable than male mortality rates. In result, male and female mortality rates are modelled by separately.

The BIC values and ranking for male and female Turkish mortality data are given in Table 4. According to Table 4, Model 8 is the best model for Turkish female mortality data and Model 4 is the best model for Turkish male mortality data. For the sake of comparison MAPE and Unexplained Variance (UV) values are given in Table 5.

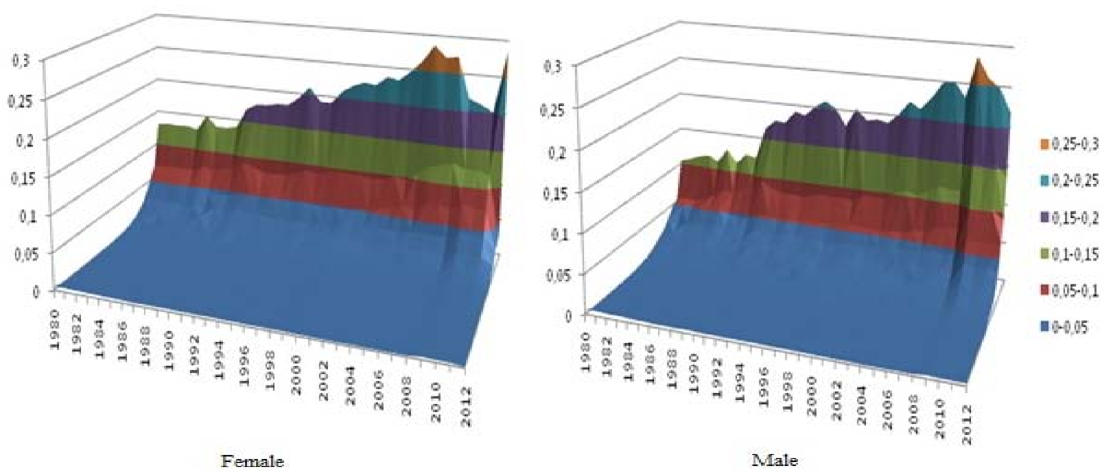


Fig. 1 Mortality rates for Turkish mortality data.

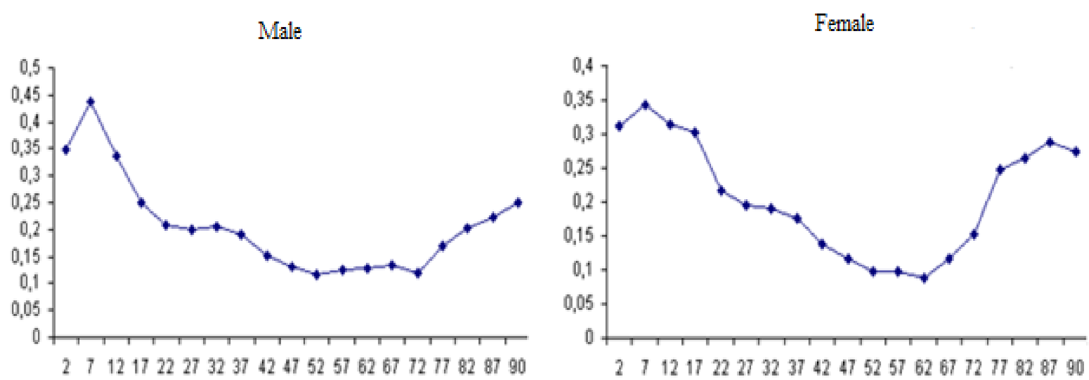


Fig. 2 Coefficient of variation of mortality rates.

Table 4 BIC values and ranking.

MODEL	Female		Male	
	BIC	Ranking	BIC	Ranking
Model 1	-8458,30	7	-6996,13	8
Model 2	-8278,93	5	-6546,25	2
Model 3	-8379,20	6	-6865,53	6
Model 4	-8563,37	8	-6458,10	1
Model 5	-8126,40	4	-6721,44	4
Model 6	-8006,23	3	-6813,75	5
Model 7	-7902,10	2	-6924,42	7
Model 8	-7864,70	1	-6586,72	3

Table 5 MAPE and variance.

MODEL	Female		Male	
	MAPE	UV	MAPE	UV
Model 1	0,1278	0,0251	0,1035	0,0023
Model 2	0,1572	0,0028	0,1397	0,0013
Model 3	0,1622	0,0013	0,1138	0,0009
Model 4	0,1567	0,0010	0,0942	0,0008
Model 5	0,1146	0,0863	0,1121	0,0014
Model 6	0,1710	0,1225	0,1342	0,0012
Model 7	0,1124	0,0395	0,1402	0,0027
Model 8	0,1048	0,0403	0,1572	0,0094

4. Results and Discussion

Mortality forecasts are playing an important role for Demography and Actuarial Science. Many stochastic mortality modelling methodologies are developed in time. In this paper, eight stochastic mortality models for central mortality rates or crude mortality rates are compared for Turkish mortality data.

When modeling and projection of mortality rates one question is how long period that should be used in. looking at Turkey mortality statistics, no mortality data available before 1980.

There are strengths in the Lee-Carter model that we could see. Lee-Carter model gave poor fit to both Turkish male amd female mortality data [1]. One reason is that trend in mortality is not linear. Another is there are different pattern effects in mortality rates.

Plat model [15] gave a comparatively good fit for Turkish female mortality data. On the other hand Renshaw and Haberman [12] is the best model for Turkish mortality data. Parameter estimates are given in Figure 3 and Figure 4.

In Turkish population, the male population is more homogenous than the male. It is found that time and cohort effects are important for male and female mortality rate modeling and projection. Life expectancy at birth for Lee-Carter model and best fit model are given in Table 6.

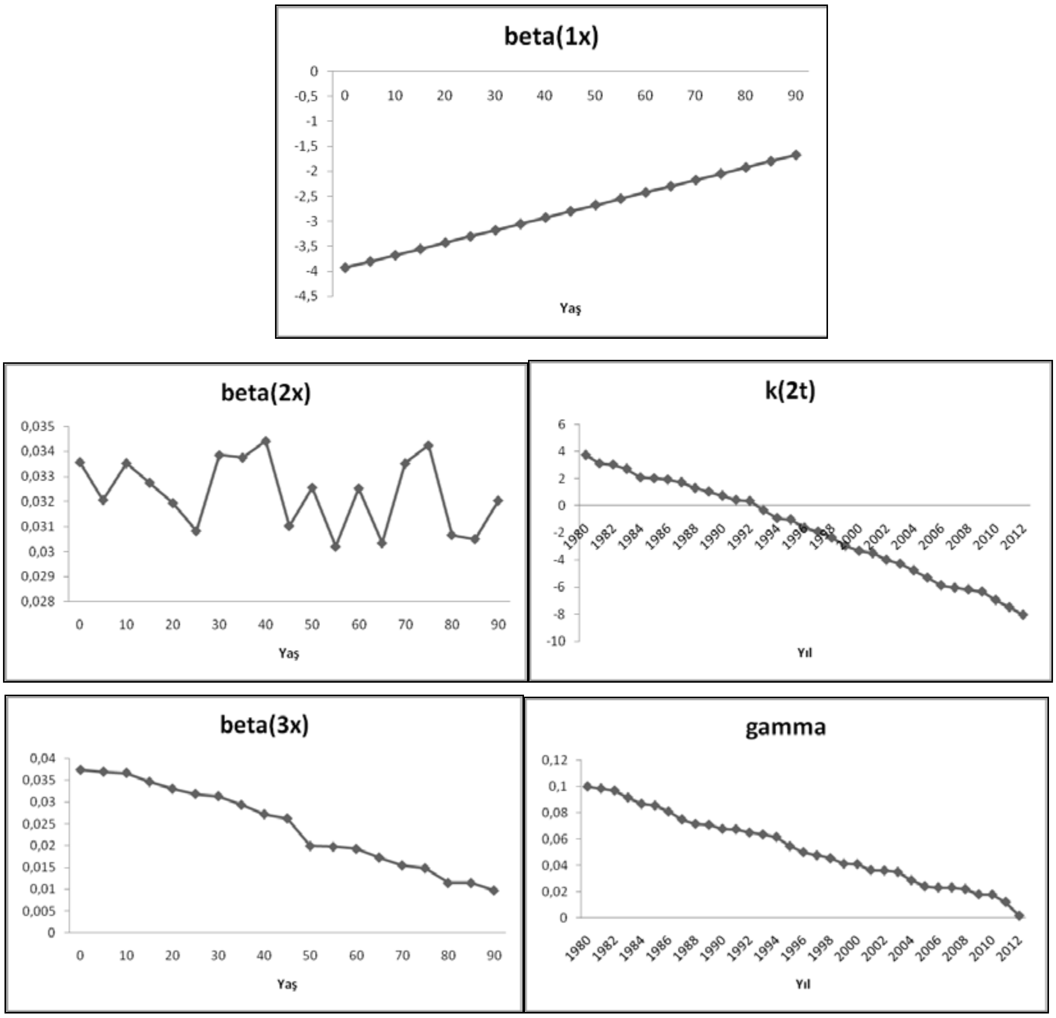
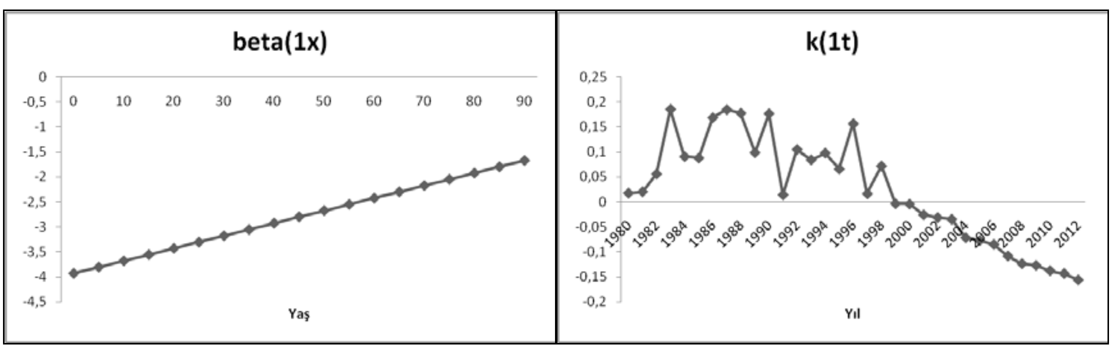


Fig. 3 Parameters for male mortality rates.



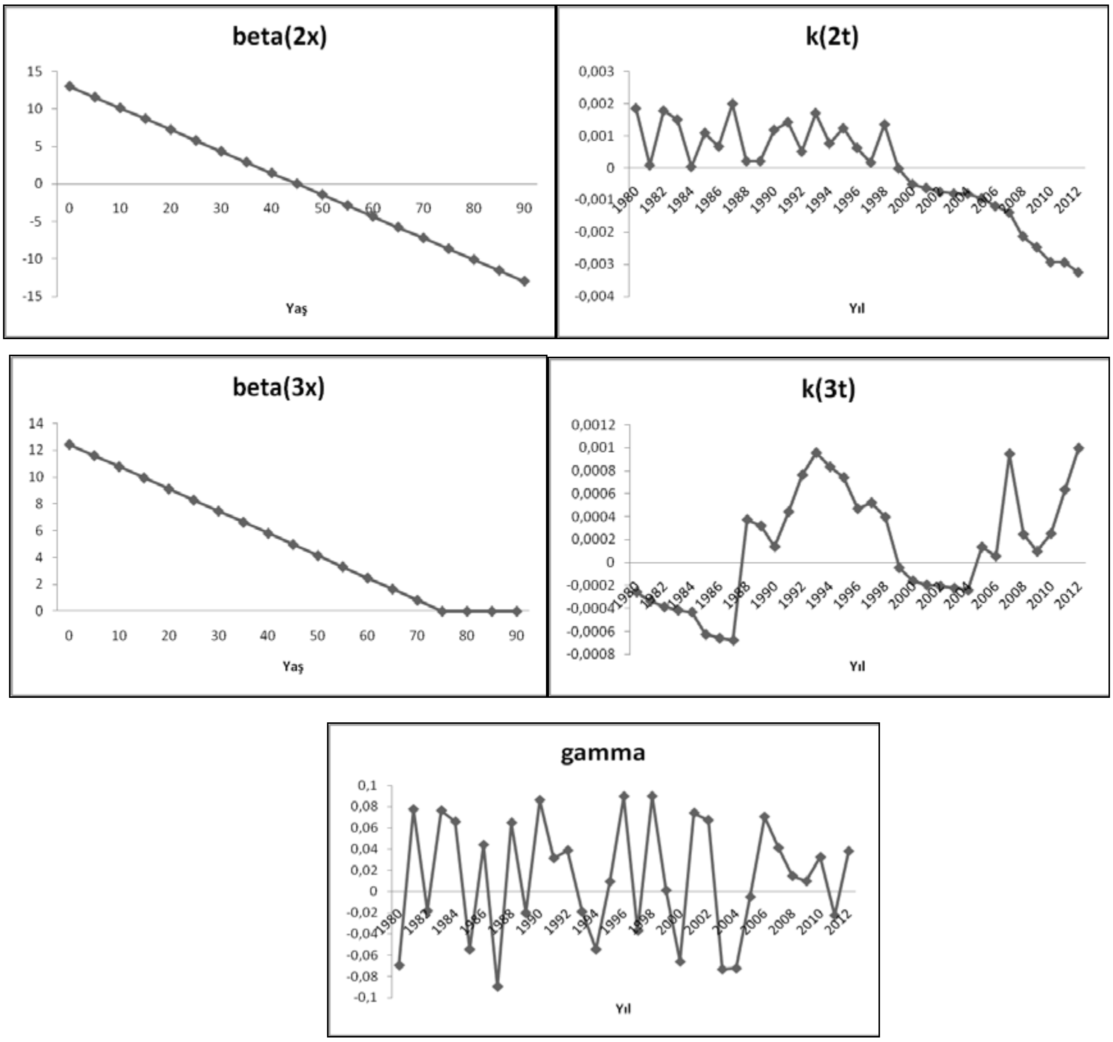


Fig. 4 Parameters for female mortality rates.

Table 6 Life Expectancy at Birth For Turkey.

Year	Female		Male	
	LC	P	LC	RH
1980	67,95	68,15	62,41	63,05
1985	68,58	69,08	63,14	63,98
1990	69,20	70,01	63,87	64,9
1995	69,83	70,93	64,6	65,83
2000	70,45	71,85	65,33	66,75
2005	71,08	72,78	66,05	67,68
2010	71,70	73,71	66,78	68,63
2015	72,33	74,63	67,51	69,53
2020	72,95	75,56	68,24	70,47
2025	73,58	76,48	68,97	71,38
2030	74,20	77,41	69,70	72,31

References

[1] R.D. Lee, L. Carter, Modeling and forecasting the time series of US mortality, Journal of the American Association 87 (1992) 659-671.

[2] A.J.G. Cairns, D. Blake, K. Dowd, G.D. Coughlan, D. Epstein, A. Ong, I. Balevich, A quantitative comparison of stochastic mortality models using data from England & Wales and the United States, The Pensions Institute 13 (1) (2007) 1-35.

[3] B. Gompertz, On the Nature of the Function Expressive of the Law of Human Mortality etc., Phil. Trans. Roy. Soc.115 (1825) 513-585.

[4] W.M. Makeham, On the law of mortality, Journal of the Institute of Actuaries 8 (1867) 301-310.

[5] L. Heligman., J.H. Pollard, The age pattern of mortality, Journal of the Institute of Actuaries 107 (1980) 49-80.

[6] M.C. Koissi, A.F. Shapiro, *The Lee-Carter Models Under the Condition of Variables Age-Specific Parameters*, Presented at the 43rd Actuarial Research Conference, 2008.

[7] F. Kul, Modeling and projection of changes in mortality structure, Doctoral dissertation, Department of Actuarial Science, Hacettepe University, Ankara, Turkey, 2015.

[8] S. Haberman, A.E. Renshaw, On age-period-cohort parametric mortality rate projections, Insurance: Mathematics and Economics 45 (2009) 255-270.

[9] H. Booth, L. Tickle, Mortality modeling and forecasting: A review of methods, ADSRI Working Paper, No. 3, 2008.

[10] A.E. Renshaw, S. Haberman, On the forecasting of mortality reduction factors, Insurance: Mathematics and Economics 32 (2003) 379-401.

[11] I.D. Currie, Smoothing and forecasting mortality rates with P-splines, Talk given at the Institute of Actuaries, 2006.

[12] A.E. Renshaw, S. Haberman, Cohort based extension to the Lee–Carter model for mortality reduction factors, Insurance: Mathematics and Economics 38 (3) (2006) 556-570.

[13] A.J.G. Cairns, D. Blake, K. Dowd, Pricing death: Frameworks for the valuation and securitization of mortality risk, ASTIN Bulletin 36 (2006) 79-120.

[14] A.J.G. Cairns, D. Blake, K. Dowd, A Two-Factor Model for Stochastic Mortality with Parameter Uncertainty: Theory and Calibration, Journal of Risk and Insurance 73 (2006) 687-718.

[15] R. Plat, On stochastic mortality modeling, Insurance: Mathematics and Economics 45 (3) (2009) 393-404.

[16] A.J.G. Cairns, D. Blake, K. Dowd, G.D. Coughlan, D. Epstein, M. Khalaf-Allah, Mortality density forecasts: an analysis of six stochastic mortality models, Insurance: Mathematics and Economics 48 (2011) 355-367.