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A Remark on Option Prices with Call Prices

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Abstract: There exist several methods how more general options can be priced with call prices. In this short note we extend these results to cover a wider class of options and market models. In particular, we introduce a new pricing formula which can be used to price more general options if prices for call options and digital options are known for every strike price. We also derive similar results for barrier type options. As a consequence, we obtain a static hedging for general options in the general class of models. Our result can be utilized in several significant applications. As a simple example, we derive an upper bound for the value of a general American option with convex payoff and characterize conditions under which the value of this option equals to the value of the corresponding European option.

Key words: Option valuation, barrier options, call options, American options, static hedging.

1. Introduction

A major part of the research about valuating option concentrates on convex payoff function, and especially the call option is widely studied in the literature. On the other hand, prices for call options can be used to valuate more complicated options. The first study on the relation between call options and general options is by Breeden and Litzenberger [1] who showed that the second derivative of a price of European call with respect to its strike is the pricing density for more general options provided that the second derivative exists. To present their result in our notation, denote by $\lambda_0(a)$ the price of a call option with strike a and assume that the second derivative $\lambda''_0(a)$ exists. Then the price of European option $f(X_T)$ is given by

$$V_o^f = \int_0^\infty f(a)\lambda_0''(a)da. \tag{1.1}$$

Bick [2] extended this result to a case where either the payoff function or the price of a call has continuous second derivative with respect to its strike price except in a finite set of points $(S_k)_{k=0}^N$ in which the left- and right derivatives exists and are finite. In particular, Bick showed that

$$\begin{aligned} V_o^f &= \mathrm{B}_T^{-1} f(0) + \int_0^\infty f''(a)\lambda_0(a)da \\ &+ \mathrm{B}_T^{-1}\sum_{k=0}^N \Delta - f(s_k)Q(X_T \geq s_k) \\ &+ \mathrm{B}_T^{-1}\sum_{k=0}^N \Delta + f(s_k)Q(X_T > s_k) \\ &+ \sum_{k=0}^N (f'(s_k+) - f'(s_k-))\lambda_0(s_k) \end{aligned} \tag{1.2}$$

where B_T denotes the bond function, Q denotes the pricing measure, $\lambda_0(a)$ is the discounted value of the call option $(X_T - a)^+$, $\Delta - f(s_k) = f(s_k) - f(s_k -)$ is the jump on the left and $\Delta + f(s_k) = f(s_k +) - f(s_k)$ the jump on the right, respectively.

For studies on the relation between call options and general options, also see Jarrow [3], who derived a characterization theorem for the distribution function of the underlying asset, and Brown and Ross [4], who consider a model with finite state space and showed

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that a wide class of options are a portfolio of call options with different strike prices. Similarly, Cox and Rubinstein [5] introduced a method to approximate continuous functions with piecewise linear functions, which are a portfolio of call options with different strikes. They also considered the pricing error of this approximation, and suggested that one should find an approximation which is the best in the sense of maximum absolute difference. However, this may cause problems when considering infinite state space.

In this article we extend the mentioned results to cover wider class of payoff functions and market models. More precisely, as our first result we show that if the payoff function is continuously differentiable except on at most countable set of points $(S_k)_{k=0}^N$ in which the left- and the right limits exists and are finite, the discounted value V_t^f at time t of a European option with payoff function $f(X_T)$ is given by

$$\begin{aligned} V_t^f &= \mathrm{B}_T^{-1} IE_Q\left[f(X_T)\middle|\Im_t\right] \\ &= \mathrm{B}_T^{-1} f(0) - \int_0^\infty f'(a)\lambda_t(da) \\ &+ \mathrm{B}_T^{-1}\sum_{k=0}^{N} \Delta - f(s_k)Q(X_T \geq s_k|\Im_t) \\ &+ \mathrm{B}_T^{-1}\sum_{k=0}^{N} \Delta + f(s_k)Q(X_T > s_k|\Im_t) \end{aligned} \tag{1.3}$$

where $\lambda_t(a)$ denotes the discounted value of the call option $(X_T - a)^+$ to time t. We also provide a similar pricing formula for barrier type options $f(X_T)1_{Y\in C}$, where Y is a random variable (e.g. Y can represent the supremum of the underlying process on $[0, T]$) and C is a Borel set.

Compared to the existing literature, many of the existing studies consider cases where either the state space is finite or the time period is discrete. Our results cover a wider class of options and models than the results of the mentioned articles. Moreover, our results are robust in the sense that we do not assume specific dynamics for the underlying asset or we do not pose any extra assumptions on the state space or time space. As such, the result applies both discrete and continuous cases. In particular, the only assumption we need is that the pricing is done by taking expectation with respect to some measure Q which is equivalent to the fact that the model is, to some extent, free of arbitrage. For details in the mathematics of arbitrage, we refer to [6] and [7].

While our result may not be the best option for pricing in all the cases and is more of a theoretical nature, it has several important applications. For example, it has been applied successfully to provide a new approach and to generalize the existing results on the rate of converge of option prices [8]. Moreover, similar techniques has been applied later in [9] and [10] to study pricing formulas for options depending on multiple underlying assets; a topic which is not widely studied in the literature. Finally, as a simple application we give analogous result for convex payoff functions and derive relatively simple but, to the best of our knowledge, new results on the values of American options.

Analytical upper bounds for prices of American options have been developed first by Chen and Yeh [11], then extended by Chang and Chung [12]. However, they do not discuss the properties of the payoff function itself but instead give conditions on the value process. In this paper we answer to the following questions:

(1) under what conditions on the payoff f and the underlying asset X_t, the price of an American option $f(X_t)$ with maturity T equals to the price of a European option $f(X_t)$ with maturity T?

(2) if the prices are not equal, what is the difference between the prices and which factors have influence on the difference?

It turns out that the answer to both questions are determined by the behaviour of the convex function f at the origin whenever the discounted asset process is a submartingale. In this article, we derive an analytical upper bound

$$V_t^{f,A} \le V_t^f + f(0) + (B_t^{-1} - B_T^{-1}) + f'_+(0) - \left(\overline{X}_t - IE_Q\left[\overline{X}_T \middle| \Im_t\right]\right) \quad (1.4)$$

where $A_+ = \max(0, A), B_- = \min(0, B)$, $\overline{X}_t$ is the discounted underlying asset, and $V_t^{f,A}$ and V_t^f denote the discounted values of American and European options. While this bound can be very inaccurate in some cases, it comes especially good if the interest rates are relatively small. This is rather surprising, since one would expect that American options are more valuable because of the possibility to optimal exercising, not because of the time value of capital arising from non-trivial bonds.

The rest of the paper is organized as follows. In section 2 we introduce the notation and present the new pricing identities, and in section 3 we derive the upper bound for the value of American options. All proofs are postponed to section 4.

2. Pricing Formula

Let S_t denote the stock price process (or S_t^k in case of several risky assets) and X_t the underlying asset of an option. As examples, X_t can be a functional of S_t like the average $X_t = \frac{1}{t}\int_0^t S_u du$ representing Asian option, $X_t = \sup_{u \le t} S_u$ representing Lookback option, $X_t = \max_{1 \le k \le d} S_t^k$ representing Rainbow option or $X_t = \sum_{k=1}^d \alpha_k S_t^k$ representing Basket option. Throughout the article, $\overline{X}_t$ denotes the discounted value of X_t i.e. $\overline{X}_t = B_t^{-1} X_t$, where the bond is given by an non-decreasing function B_t with B_0=1. The expectation with respect to risk-neutral measure Q is denoted by IE_Q. In general, we also use the terms “positive” and “increasing” as synonyms for “non-negative” and “non-decreasing”. Similarly, the symbol IR_+ refers to positive real numbers including zero. We also use term *the value at time t* of an option *f*(*X*) with maturity *T* as synonym for the discounted value at time *t*, and this value is denoted by V_t^f (or $V_t^{f,A}$ for American options, respectively). We omit the dependence on the measure *Q* on the notation despite the fact that the measure is not necessarily unique. We also omit *t* on the notation whenever we consider the price of the option, i.e. the value at time $t = 0$.

We assume that on a certain market model, we are given the underlying asset X_t and the equivalent martingale measure *Q*. We consider the following class of payoff functions.

Definition 2.1.

For a function $f : IR_+ \to IR$, we denote $f \in II_Q(X_T)$ if the following conditions are satisfied:

(1) f is continuously differentiable except on at most countable set of points $0 < s_1 < s_2 < ... < s_N$ (and possibly on $s_0 = 0$) in which f and f' have jump-discontinuities,

(2) $f(X_T) \in L^1(Q)$,

(3) f satisfies

$$\lim_{x \to \infty} \left| f(x-) \right| Q(X_T \ge x) = 0 \quad (2.1)$$

and,

(4) the stochastic Riemann-Stieltjes integral

$$\int_0^\infty f'(a) \lambda_t(da)$$

exists and is finite for every $t \in [0,T]$, where

$$\lambda_t(a) = IE_Q\left[B_T^{-1}(X_T - a)_+ \middle| \Im_t\right] \quad (2.2)$$

denotes the value of the European call option at t with strike a.

Remark 2.1.

Note that the technical assumptions are not very restrictive. Indeed, the function $\lambda_t(a)$ is absolutely continuous and f' has only jump discontinuities. Thus the Riemann-Stieltjes integral is well-defined on every interval [*0*, *N*]. Moreover, it coincides with the Lebesgue integral, and the derivative of $\lambda_t(a)$ is the value of a negative digital option with payoff $-1x \geq a$. Note also that condition (2.1) implies that $\lim_{x\to\infty} |f(x-)| Q(X_T \geq x | \Im_t) = 0$ (in probability) for every $t \in [0,T]$. Moreover, by examining the proof of Theorem 2.1 we obtain that conditions (2), (3) and (4) are closely related: if the sums of the jump-terms are finite then the limit in (2.1) is finite if and only if the Riemann-Stieltjes integral is finite and these follows from assumption (2).

For Barrier type options, we consider the class $II_Q(X_T, Y, C)$ and denote $f \in II_Q(X_T, Y, C)$, if:

(1) the assumptions (1) and (2) of the class $II_Q(X_T)$ are satisfied,

(2) $\lim_{x\to\infty} |f(x-)| Q(X_T \geq x, Y \in C) = 0$

and,

(3) the stochastic Riemann-Stieltjes integral

$$\int_0^\infty f'(a) \lambda_t^{Y,C}(da)$$

exists and is finite for every $t \in [0,T]$, where

$$\begin{aligned} &\lambda_t^{Y,C}(a) \\ &= IE_Q\left[\mathrm{B}_T^{-1}(X_T - a) + 1_{Y\in C} \middle| \Im_t\right] \end{aligned} \quad (2.3)$$

Remark 2.2.

Note that we have $II_Q(X_T) \subset II_Q(X_T, Y, C)$ for every *Y* and measurable set *C*.

Now we can state our main theorems:

Theorem 2.1.

Let $f \in II_Q(X_T)$. Then the value of an option $f(X_T)$ at time t is given by

$$\begin{aligned} V_t^f &= \mathrm{B}_T^{-1} f(0) - \int_0^\infty f'(a)\lambda_t(da) \\ &+ \mathrm{B}_T^{-1} \sum_{k=0}^{N} \Delta - f(s_k) Q(X_T \geq s_k | \Im_t) \\ &+ \mathrm{B}_T^{-1} \sum_{k=0}^{N} \Delta + f(s_k) Q(X_T > s_k | \Im_t) \end{aligned} \quad (2.4)$$

where $\Delta - f(s_k) = f(s_k) - f(s_k -)$ and $\Delta + f(s_k) = f(s_k +) - f(s_k)$. The jump from the left at zero is defined as $\Delta - f(0) = 0$.

Proof. Without loss of generality and for the sake of simplicity, we assume in that $B_t = 1$, $t \in [0,T]$, since the result for non-constant deterministic bond B_t follows easily with obvious modifications. We also prove formulas only for $t = 0$ as the general case follows simply by considering conditional expectations and the same arguments can be applied.

Consider first a payoff of form $g(x) = f(x) 1_{\alpha \leq x \leq \beta}$ with some $\alpha \geq 0,\ \alpha < \beta < \infty$, where f is continuous on $[\alpha, \beta]$, we can approximate it with

$$\begin{aligned} g_n(x) &= f(\alpha) 1_{x=a} \\ &+ \sum_{k=1}^{n} (c_k x + b_k) 1_{a_k < x \leq a_{k+1}} \\ &= f(\alpha) 1_{x=a} \\ &+ \sum_{k=1}^{n} (c_k x + b_k)\left(1_{a_k < x} - 1_{a_{k+1} < x}\right) \end{aligned} \quad (2.5)$$

where $\alpha = a_1 < a_2 < \ldots < a_{n+1} = \beta$ is a partition of the interval $[\alpha, \beta]$, and the coefficients are given by

$$c_k = \frac{f(a_{k+1}) - f(a_k)}{a_{k+1} - a_k},$$

and

$$b_k = f(a_{k+1}) - c_k a_{k+1} = f(a_k) - c_k a_{k1}.$$

The payoff of a call-option with strike *K* is given by

$$p(x, K) = (x - K)^+ = x1_{x>K} - K1_{x>K}.$$

Simple computations yields

$$\begin{aligned} &g_n(x) \\ &= f(\alpha)1_{x \ge \alpha} - f(\beta)1_{x>\beta} \\ &+ \sum_{k=1}^{n} c_k \left[p(x, a_k) - p(x, a_{k+1}) \right]. \end{aligned} \tag{2.6}$$

By taking expectation with respect to the equivalent martingale measure *Q*, we obtain

$$\begin{aligned} V^{g_n} &= f(\alpha) Q(X_T \ge \alpha) - f(\beta) Q(X_T > \beta) \\ &+ \sum_{k=1}^{n} c_k \left[\lambda(a_k) - \lambda(a_{k+1}) \right] \end{aligned}$$

Now g_n converges to g pointwise and by mean value theorem,

$$\sum_{k=1}^{n} c_k \left[\lambda(a_k) - \lambda(a_{k+1}) \right] \to -\int_0^b f'(a) \lambda(da)$$

as n tends to infinity. Applying dominated convergence theorem yields

$$\begin{aligned} V_t^g &= \mathrm{B}_T^{-1} f(\alpha) Q(X_T \ge \alpha | \Im_t) \\ &- \mathrm{B}_T^{-1} f(\beta) Q(X_T > \beta | \Im_t) \\ &- \int_\alpha^\beta f'(a) \lambda_t(da) \end{aligned} \tag{2.7}$$

Similarly, for the payoff function of form $g^0(x) = f(x)1_{\alpha<x<\beta}$ we get

$$\begin{aligned} V_t^{g^0} &= \mathrm{B}_T^{-1} f(\alpha+) Q(X_T > \alpha | \Im_t) \\ &- \mathrm{B}_T^{-1} f(\beta-) Q(X_T \ge \beta | \Im_t) \\ &- \int_\alpha^\beta f'(a) \lambda_t(da) \end{aligned} \tag{2.8}$$

Now to obtain the general case, we put $g_b(x) = f(x)1_{0 \le x < b}$ where $s_{n+1} = b$, and write

$$\begin{aligned} g_b(x) &= \sum_{k=0}^{n} f(x) 1_{s_k < x < s_{k+1}} \\ &+ \sum_{k=0}^{n} f(x) 1_{x = s_k} \end{aligned}$$

For terms on the first sum we obtain by applying (2.8) that

$$\begin{aligned} V_t^{g_b} &= \mathrm{B}_T^{-1} f(o) - \int_0^b f'(a) \lambda_t(da) \\ &+ \mathrm{B}_T^{-1} \sum_{k=0}^{n} \Delta - f(s_k) Q(X_T \ge s_k | \Im_t) \\ &+ \mathrm{B}_T^{-1} \sum_{k=0}^{n} \Delta + f(s_k) Q(X_T > s_k | \Im_t) \\ &- \mathrm{B}_T^{-1} f(b-) Q(X_T \ge b | \Im_t). \end{aligned}$$

Letting $b \to \infty$, applying dominated convergence theorem and taking account the assumptions (3) and (4) on $II_Q(X_T)$ we obtain the result.

□

Example 2.1.

As a non-trivial example, consider a power call option for which the payoff is given by $f(x) = (x^n - K)^+$ for some integer *n*. In this case the value of $f(X_T)$ at *t* is given by

$$V_t^f = \int_{K^{\frac{1}{n}}}^{\infty} n a^{n-1} \lambda_t(da)$$

Example 2.2.

Consider spread-option of type $(S_T^2 - S_T^1)^+$. By considering random variable $X_T = S_T^2 - S_T^1$ we obtain that the price is given by

$$V^{spread} = \int_0^\infty Q(S_T^2 > a + S_T^1) da$$

Remark 2.3.

Since $\lambda_t(a)$ is absolutely continuous with respect to a, we have

$$\int_0^\infty f'(a)\lambda_t(da) = -\int_0^\infty f'(a)B_T^{-1}Q(X_T > \alpha|\Im_t)da. \tag{2.9}$$

Hence the value process V_t of a portfolio

$$f(0) + \int_0^\infty f'(a)I_{X_T>a}da + \sum_{k=0}^N \Delta - f(s_k)I_{X_T} \geq s_k + \sum_{k=0}^N \Delta - f(s_k)I_{X_T} > s_k.$$

equals to the value process V_t^f of $f(X_T)$ almost surely for every *t*. Hence we obtain a static hedging for $f(X_T)$ if we have access to digital options for every strike price. Of course, it is not a realistic assumption to have access to digital options with all strikes. However, this result can be used to construct approximations of hedges. For application to numerical approximations, we refer to [8]. For more information on static hedging which is topic if increasing popularity, we refer to Carr and Picron [13] and references therein.

Remark 2.4.

Note that if the derivative f' is absolutely continuous on every interval (s_k, s_{k+1}), then the second derivative of f exists for almost every a and integration by parts gives the formula (1.2). In this case we find a static hedging strategy by investing to call options too. Similarly, if $Q(X_T \geq a|\Im_t)$ is absolutely continuous, integration by parts gives (1.1).

Remark 2.5.

If f is a linear combination of convex functions, integration by parts yields

$$V_t^f = B_T^{-1}f(0) + f_+'(0)IE_Q\left[\overline{X}_T|\Im_t\right] + \int_0^\infty \lambda_t(a)\mu(da) \tag{2.10}$$

where μ is the measure associated with the second derivative of f. This also follows directly from the well-known representation [14]

$$f(x) = f(0) + f_+'(0)x + \int_0^\infty (x-a)^+ \mu(da) \tag{2.11}$$

for convex functions $f: IR_+ \to IR.$

The following extension to Barrier type options follows with the same lines.

Theorem 2.2.

Let $f \in II_Q(X_T, Y, C).$ Then the value $V_t^{f,Y,C}$ of the barrier option $f(X_T)1_{Y\in C}$ at time *t* is given by

$$\begin{aligned} &V_t^{f,Y,C} \\ &= B_T^{-1}f(0)Q(Y\in C|\Im_t) \\ &-\int_0^\infty f'(a)\lambda_t^{Y,C}(da) \\ &+B_T^{-1}\sum_{k=0}^N \Delta - f(s_k)Q(X_T \geq s_k, Y\in C|\Im_t) \\ &+B_T^{-1}\sum_{k=0}^N \Delta + f(s_k)Q(X_T > s_k, Y\in C|\Im_t). \end{aligned} \tag{2.12}$$

Remark 2.6.

Note that we can obtain formulas corresponding to (1.2) and (2.10) for barrier type options as well.

Remark 2.7.

In both of our main theorems, we assumed that the bond is a deterministic function. However, we may allow the bond to be an adapted, increasing process with obvious modifications in the theorems.

3. Upper Bound for American Option Values

In this section we give a simple application and study American options with convex payoff function f. We do not restrict the function f to be positive and it may not be clear what is the interpretation of American option with negative payoff f. However, mathematically we can treat such payoff functions, and a possible financial interpretation for negative values is that we have an agreement where we are determined to pay some amount of money, but we can influence on the amount of losses (by stopping). For example, this could represent a situation for an insurance company which has an option to revoke a contract if the losses seem to increase too much in a simplified model where the payments are gathered only in the beginning of contract.

Proposition 3.1.

Assume that the payoff function f is convex and assume that the discounted underlying asset $\overline{X}_t$ is a submartingale. Then the value of an American option $f(X_t)$ has an upper bound

$$\begin{aligned} V_t^{f,A} &\leq V_t^f + f(0) + (B_t^{-1} - B_T^{-1}) \\ &+ f_+^{'}(0) - (\overline{X}_t - IE_Q[\overline{X}_T|\Im_t]). \end{aligned} \tag{3.1}$$

where $A+ = \max(0, A)$ and $B- = \min(0, B)$.

Proof. If the discounted underlying asset $\overline{X}_t$ is a submartingale, then the value at time t of an American call option $(X_t - K)^+$ with terminal time T is the same as the value at time t a European call option $(X_T - K)^+$. Consequently, using Tonelli's Theorem and representation (2.11) we obtain

$$\begin{aligned} &\sup_{r\in[t,T]} IE_Q[B_r^{-1} f(X_r)|\Im_t] \\ &\leq \sup_{r\in[t,T]} IE_Q[B_r^{-1} f(0)|\Im_t] \\ &+ \sup_{r\in[t,T]} IE_Q[f_+^{'}(0) B_r^{-1} X_r|\Im_t] \\ &+ \sup_{r\in[t,T]} \int_0^\infty IE_Q[B_r^{-1}(X_r - K)^+|\Im_t]\mu(da) \\ &\leq \sup_{r\in[t,T]} IE_Q[B_r^{-1} f(0)|\Im_t] \\ &+ \sup_{r\in[t,T]} IE_Q[f_+^{'}(0) B_r^{-1} X_r|\Im_t] \\ &+ \int_0^\infty \sup_{r\in[t,T]} IE_Q[B_r^{-1}(X_r - K)^+|\Im_t]\mu(da) \\ &= \sup_{r\in[t,T]} IE_Q[B_r^{-1} f(0)|\Im_t] \\ &+ \sup_{r\in[t,T]} IE_Q[f_+^{'}(0) \overline{X}_r|\Im_t] \\ &+ \int_0^\infty IE_Q[B_T^{-1}(X_T - K)^+|\Im_t]\mu(da). \end{aligned}$$

Thus, by using (2.10), we obtain

$$\begin{aligned} &V_t^{f,A} - V_t^f \\ &\leq \sup_{r\in[t,T]} IE_Q[B_r^{-1} f(0)|\Im_t] - B_T^{-1} f(0) \\ &+ \sup_{r\in[t,T]} IE_Q[f_+^{'}(0)\overline{X}_r|\Im_t] \\ &- IE_Q[f_+^{'}(0)\overline{X}_T|\Im_t]. \end{aligned}$$

Evidently,

$$\begin{aligned} &\sup_{r\in[t,T]} IE_Q[B_r^{-1} f(0)|\Im_t] - B_T^{-1} f(0) \\ &= f(0) + (B_t^{-1} - B_T^{-1}). \end{aligned}$$

Similarly, by the submartingale property of $\overline{X}_t$, we obtain

$$\begin{aligned} &\sup_{r\in[t,T]} IE_Q[f_+^{'}(0)\overline{X}_r|\Im_t] \\ &- IE_Q[f_+^{'}(0)\overline{X}_T|\Im_t] \\ &= f_+^{'}(0) - (\overline{X}_t - IE_Q[\overline{X}_T|\Im_t]). \end{aligned}$$

□

Remark 3.1.

The result can be generalised to a case where the bond is an adapted, increasing process B_t with $B_0 = 1$ instead of deterministic function. In this case, we have to replace the term B_T^{-1} on the upper bound by conditional expectation $IE_Q\left[B_T^{-1}\middle|\Im_t\right]$.

The upper bound becomes impractical if the ratio

$$\frac{f(0)+(B_t^{-1}-B_T^{-1})+f_+^{'}(0)-(\overline{X}_t-IE_Q\left[\overline{X}_T\middle|\Im_t\right])}{V_t^f} \quad (3.2)$$

is large and in many cases the upper bound is not very accurate. For example, the upper bound for the price for put option is inefficient if the option is out of the money. The benefit of the result is that the upper bound can be useful especially in cases where the interest rate is low, and then the American option is not much valuable than its European counterpart. This suggests that it is more important to have possibility for early exercise because of interest rates while intuitively one could think that American options are more valuable because of the possibility to exercise at the optimal moment. Moreover, as a simple corollary we obtain a useful result which gives sufficient conditions under which the values of European and American options are equal. The result in a case where X_t is the stock itself (and hence a martingale) and $f(0)=0$ is already proved in [15].

Corollary 3.1

Assume tha f is convex and $\overline{X}_t$ is a submartingale. Then the value of an American option $f(X_t)$ at time t equals to the value of its European counterpart at time t if the following two conditions hold:

(1) bond B_t is a constant *1* or $f(0) \le 0$,

(2) discounted underlying asset $\overline{X}_t$ is a martingale or $f_+^{'}(0) \ge 0$.

Note that if f is not linear and the process X_t has support on all of (0, ∞) (e.g., geometric Brownian motion), then the Jensen's inequality is strict. This implies that the optimal moment to exercise the option is the maturity T. We also note that it is somewhat common in the literature to assume interest rates to be zero for simplicity. The result presented here indicates that this assumption should never be done while studying American options. For example, the values of American and European put options are equal if the bond is not present.

We have analogous results for options with concave payoff function and the proofs are based on the fact $-\inf(-h) = \sup h$ and the arguments in the proof of Proposition 3.1. In this case, we obtain the conditions under which the value is the discounted instruction value $B_t^{-1} f(X_t)$. Moreover, if the support of X_t is the whole positive real line, one should exercise the option immediately.

Example 3.1 Consider a power option of example 2.1. By Corollary 3.1, the values of American and European options are the same.

Example 3.2 Consider a market model with n risky assets $S_t^1,\ldots,S_t^n$ where $\overline{S}_t^k$ is a martingale with respect to Q for every k and consider American Basket option and American Rainbow option that is based on the stock with the best performance. For the first one, $\overline{X}_t = \sum_{k=1}^n \alpha_k \overline{S}_t^k$, which is clearly a martingale. For the second one, $\overline{X}_t = \max_{1\le k\le n} \overline{S}_t^k$, which is a submartingale (see [16]). Hence we may apply Corollary 3.1 to obtain conditions under which the values of American type Basket or Rainbow options equal to their European counterparts. To the best of our knowledge, this is not mentioned in the literature.

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