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Opening Problems Is One Step forward to Reach More Students

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Abstract: To characterize the quality of mathematics teaching in a country is not an easy task. Outside Hungary you may hear very often that "Teaching problem-solving in Hungarian mathematics education is world-famous thanks to George Pólya". Maybe people have also heard about some excellent Hungarian mathematicians like J. Neumann, P. Halmos, E. Szemerédi, L. Lovász. It is true that these refer to the excellence of mathematics teaching in Hungary, but do not say anything about the real situation of Hungarian mathematics teaching today. To characterize it we will analyze in details the PISA 2012 mathematics test results and university mathematics tests that students have to sit at the beginning of the higher education studies. Based on these we narrow our attention to students who continue their studies at universities or colleges. This is usually the top 20%-25% of a year group. The article tries to answer the question: How can we prepare these students more effectively for their higher level studies and for their future jobs? Our statement is: opening problems is one step forward to reach more students who can then become more successful in mathematical problem-solving. After some theoretical considerations we analyze two experiments: one classroom experiment and one focusing individually on not highly talented students. Finally, we summarize some suggestions about how to spread the research results into the mainstream of Hungarian mathematical problem solving teaching.

Key words: Mathematics education, working memory, cognitive load, open problems, average ability students.

1. Introduction

Assessment (Testing) is inevitably an important part of education, so of mathematics education too. The tests measure the products of teaching such as how students demonstrate their acquired knowledge, skills and problem solving abilities … etc. The results are highly dependent on the students' memory and are influenced by four factors: encoding conditions; personal, subject conditions; material, content variables; retrieval conditions. Tests can be international or central in a country and of course some are written in the process of teaching. The latest are realized usually directly after finishing teaching a topic. However, the results of post—tests and delayed tests used in mathematical—didactical research may be useful, even based on these results we cannot state much about the level of mathematics teaching in a country. In this case international tests and central tests give more information.

In the recent years, Hungarian universities and colleges have created tests in mathematics for the first year students with the help of which they try to investigate how much of the secondary mathematics contents and skills these students remember. These tests are delayed tests and their tendencies may give very useful information about secondary school mathematics teaching. In the following we analyze the results of Hungarian students on the PISA 2012 mathematics tests and on some university tests,

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furthermore, we make some comments about the Hungarian maturity exam in mathematics.

On the PISA 2012 mathematics test out of the 65 participants Hungary reached the 39th place with an average of 477 points (OECD average is 494 points). The top rating level 6 was reached only by 9.3% of the Hungarian students, and the lower levels 2 or 1 were achieved by 28.1% of them. On the creative problem solving PISA 2012 test out of the 44 participants Hungary placed 33rd. Here, 35% of the Hungarian students reached level 2 or level 1. Hungary has always participated in the PISA tests right from the beginning, usually achieving OECD average results, but our students' results show a decreasing tendency. As a reaction to these results the Hungarian government has introduced the so called competence tests for grades 4, 6, 8 and 10. These tests contain so-called PISA like problems and the schools are evaluated based on their students' results on these tests. A direct consequence of this is that a few weeks before the tests some schools start preparing their students directly for the competence tests, practicing problems taken from earlier competence tests to make the style of the problems familiar for their students. So, the results of these tests do not reflect the whole reality. The answer of a head teacher to my question: "Why are Hungarian PISA results so low?" was interesting. She answered: "If the achievement of the students on PISA tests was part of the official school evaluation, they would take these tests more seriously".

Another important test in Hungarian mathematics education is the central maturity exam at the end of year 12. The exam has two levels, middle and higher level. The higher level exam has a written and an oral part. Differential and integral calculus are only part of the higher level schemes of work and the majority of the topics is discussed more deeply in courses preparing students for the higher level exam. Most of the higher level problems are complex ones, and many of them are modeling problems while the middle level problems usually include basic mathematical tasks, algorithms or procedures. In one year about 95,000 students take the mathematics maturity exam out of whom about 3,500 students take the higher level exam (3.6%!). The problem is that most of these students are not prepared to solve complex problems. From the above mentioned data we may ask why so few students choose the higher level exam. Universities training future engineers, information technologists, mathematicians, mathematics teachers, and architects have places to offer for about 20%-25% of a year group, and only about 3.6 % of a year group takes the higher level maturity exam in mathematics, although a high level of mathematical knowledge is a requirement for these majors.

The first year university and college students have to sit a test at the beginning of their university or college studies in which the problems are based on the middle level maturity exam requirements. Based on their result most of the students must participate on an "adjustment" course where they go through the basic secondary mathematical concepts, algorithms, procedures, etc. Without going into details we mention some findings based on these tests of two elite universities, the Technical University of Budapest and the Eötvös Lóránd University of Budapest: most of the first year students do not understand relationships; do not know key ideas; have very week analyzing abilities; their work is hard to follow; their knowledge is superfluous; they have very week modeling abilities; their imaginative abilities are week too. We do not need to emphasize how important these factors are in effective problem solving. To summarize it: it seems that the whole system – the Hungarian elementary and secondary mathematics education does not work very effectively. What can we do to achieve some changes? We concentrate only on one but an important issue: how can we reach not only the top 10% of students—see PISA results above—in mathematics lessons but also "the rest", the less able ones? Our opinion is that the teaching style of mathematics that works for the top ten percent is not effective for the next 10%-20%. For example, we can mention one difficulty that is related to closed problems. In Hungarian task collections and exams most mathematics problems are posed in a closed form and a lot of students cannot start solving them without help because they mostly need to apply top-down deductive methods. For these students opening a problem gives a chance to take steps individually towards the solution, for example investigating some concrete cases, which is more of a bottom-up, inductive method. We will report about a teaching experiment in which cooperative teaching techniques and open problems were applied and we report about our experience related to not highly talented students' individual problem-solving.

2. Theoretical Background

Firstly, we concentrate on some findings of cognitive neuroscience to demonstrate how different students can be when it comes to learning mathematics and these factors should be taken into consideration in mathematics teaching. Our basic point of view is *effective mathematics teaching does note exists without taking the human cognitive architecture into consideration*. Unfortunately, the pedagogical and psychological sciences are neglected in Hungarian mathematics education, and the science mathematics plays a one-sided dominant role.

2.1 Memory Structures

Most neuroscientists accept Baddeley's model of memory structures: perceptual memory, working memory, long term memory. Here, we analyze the last two one.

Working Memory (WM). In problem solving, the role of WM is vital. It is called the workbench of our brain; it is the active problem space. It has four components: "phonological loop" to hold and rehears verbal information; "visual-spatial sketchpad" to hold and rehears visual and spatial information; "episodic buffer" which connects the verbal and visual-spatial information directed by the "central executive" with the help of the information taken from the long term memory. The central executive is the so called supervisory attention system, because it monitors and controls the information processing in our brain. Our WM constructs plans, uses transformation strategies, analogies, metaphors, brings together things in thought, abstracts and externalizes mental representations. In problem solving students need a clear mental representation of the task (Understanding the problem). While seeking a strategy (solution method), students need to hold the conditions and the goal and the possible transformation (solution) steps in their memory, and taken this into consideration they should monitor their progress in the solution, inhibit wrong, unsuccessful ideas and check their results. It is very hard to make these components appear in class teaching [1].

Executive functions**:** goal setting, planning, organizing, prioritizing, initiating, holding information, inhibiting irrelevant information, self-monitoring, memorizing, self-regulating, representing, problem solving.

Limits of WM**:** very limited capacity holding 7 ± 2 info units, time limit: 18-30 sec without rehearsal, goal maintenance, inhibiting of irrelevant information.

Long Term Memory (LTM). LTM contains information in the form of schemas. Schemas are abstract, structured, dynamic representations of information. Schema automaticity means a skill and a procedure is learned so that it does not place demand on WM which has a very important implication: we may extend the capacity of WM with recalling relevant schemas from the LTM which functions as only one information unit in the WM (See capacity limit!).

In the WM the novel information is either incorporated into existing schema(s) or similar schema(s) are produced and altered or new schema(s) are recoded back into the LTM.

2.2 Overcoming the Limits of WM: Cognitive Load Theory

Cognitive Load Theory (CLT). Cognitive load can be defined as the load imposed on the WM by information being presented. It is based on the following assumptions:

1) The capacity of WM is limited;

2) Pieces of information in the LTM are stored as schemas;

3) Schemas represent units of information;

4) Automaticity of schemas in the LTM can be achieved.

Learning requires an active conscious process in the WM.

Types of cognitive loads. Intrinsic cognitive load depends on the elements that must be processed simultaneously. For example, when solving word problems reading the problem, concluding what the problem asks and solving the problem are elements that interact. Intrinsic cognitive load embedded in the problem cannot be influenced by us, teachers. Examples: low intrinsic cognitive load: 5+6, high intrinsic load $2\frac{3}{4}+5\frac{6}{7}$.

Extraneous cognitive load. It depends on the way the information is presented. It may include superfluous information that is not necessary for learning the presented material such as background music or it holds mental representations of facts or figures. For example, this is why the fact that a geometrical figure and the corresponding statements are separated may be hard to comprehend for some students.

Germane cognitive load. It means the cognitive load placed on the WM at schema formation, integration and automation, explaining the differences between students in experience, ability level and content knowledge.

Cognitive load = intrinsic load + extraneous load + germane load.

When planning teaching we must take the possible cognitive loads into consideration.

Reduction of cognitive load. Here, we mention only the factors that are important for our experiments. For more factors, please see [2].

1) Use of sample problems with solutions. “Research has provided overwhelming evidence that, for everyone but experts, partial guidance during instruction is significantly less effective than full guidance” [2]. In our experiment when we use guiding questions it means not fully guided but strongly guided instructions;

2) Goal free problems. In some problems the distance between the starting phase and the goal is very big and students are asked to find as many pieces of data as they can. Example: “In a triangle two sides are 7 cm and 11 cm long, the angle between them is 73° . Find all the missing information about this triangle that you can”. In our experiment, opening problems goes in this direction;

3) Applying cooperative teaching techniques. Research experiments show that in group work the WM capacities of the members are added together, so the cognitive load is not very high for the individuals. Our classroom experiment is based on group work [3].

3. Experience with Some not highly Talented Students

Below I will present some of my experience with some students trying to solve the presented problems. The schools which the students attend are located in two small towns in the southern part of Hungary. The Hungarian school system is very selective, and the so called elite schools with good teachers and good students are located in Budapest and in big cities. Of course there are talented students and students with good mathematical ability in small town schools as well, but based on my experience I state that the teaching style that is beneficial for the talented students is not always useful for the good but not outstanding students. As a result, the later ones might lose their interest in learning mathematics because of failure of success. But these “good” students also want to continue their studies at universities—see the

10%-15% mentioned in introduction—so we need to educate them accordingly.

3.1 Problems taken from Hungarian Task Collections

1) The sum of three integers is 2014. Is it possible that their product is 111 111?

2) From the sum of five consecutive odd numbers we subtract the sum of even numbers lying between the odd numbers, the result is 55. Find these numbers.

3) We add three consecutive natural numbers and we add the following three consecutive natural numbers, too. Is it possible that the product of these two sums is 111 111 111?

4) Is it possible to divide six consecutive integers into two groups so that the sum of the numbers in the two groups equals?

5) Can we divide a square into exactly 2014 squares without overlapping and without gaps?

6) We added 2014 consecutive natural numbers. Is the sum divisible by 2014? Is the sum divisible by 2013?

3.2 Experience with the Students

Task 1: The sum of three integers is 2014. Is it possible that their product is 111 111?

Grade 5 (6 students)

Six students tried to solve the problem but none of them were able to start, so I modified the problem*: The sum of three integers is 10. Is it possible that their product is 27*? I also gave a hint for the students: try to find 3 numbers whose sum is 10 then calculate the product of the terms. Look for more options! Do you notice something? Justify your idea.

For 3 students the modified problem was very unusual. "What shall I do?" asked *V*. For these 3 students it was necessary to show 3 concrete numbers whose sum was 10.

Three students could find more solutions for the sum and calculated the products of those numbers, but they needed additional help with noticing a pattern: What kind of numbers are the products? Only *M* noticed the right pattern for the terms, namely that either the terms of the sum are all even numbers or two terms are odd and one is even and there are no other options. So, the product of these numbers is always even, it cannot be an odd number.

Grade 6 (*BZ*, *BT*)

There were no reactions from *BZ* and *BT* for the original version. However, for the modified problem they managed to find more numbers whose sum is 10 and they calculated the products, but they needed help with noticing a pattern and explaining what they found—"What is common in the products? Why? What kind of numbers are the terms of the products?"

Grade 7 (*KG*)

KG had no idea how to start the first version of the problem. With the modified version, for 10, he could find more cases and noticed the pattern of the terms—either 3 even or 2 odd and 1 even number—and was able to give a correct argument for what the products are like.

It is interesting to mention that the young students used only positive integers, it was necessary to ask them to choose negative integers too and they saw that the same argument also works in their case.

Grade 10 (*CD*)

He wrote 111 111 as a product of its prime factors. He obtained 111 111 = $3 \cdot 7 \cdot 11 \cdot 13 \cdot 37$ and he said: "We should arrange the five factors into 3 groups and calculate the sums of the obtained numbers and check whether it is 2014. But it is not a nice work". Then, the idea occurred: "First, try with smaller numbers!" Following this he was able to find the impossibility of the original statement with right arguments very quickly.

Grade 11 (*BD*, a very bright student)

"Of course it is not possible because the sum contains either three even or two odd and one even numbers, so the product is always an even number that is why it cannot be 111 111 and the product of numbers never can be an odd number if their sum is even".

Task 2: From the sum of five consecutive odd numbers we subtract the sum of even numbers lying between the odd numbers, the result is 55. Find these numbers.

The number 55 was too large for *BZ*, *BT* and *KG*. After it was suggested to start with small odd numbers the students concluded that the final difference is the middle odd number. For the question why they answered: “Because it works”. With help they formed odd-even pairs noticing that the differences cancel each other out. The given number must be an odd number in the middle. It works if the number of odd numbers is odd (There exists a middle odd number). What if the number of odd numbers is even? Trying with small numbers the students found that the “middle number idea” works here too. Their argument again was based on forming pairs. The older students solved this problem with using equations.

Task 3: Can we divide a square into exactly 2014 squares without overlapping and without gaps?

Nobody managed to solve the original problem. For some students the second variation was a great help in finding the solution but for other students only the third variation—using concrete materials—helped.

Second variation of the original problem: Try to divide a square into small squares without overlapping and without gaps. Start with small numbers like 2, 3, 4, 5, 6, 7, 8… Try to find a pattern! For which numbers can you make the partition? Justify your general statement! Can you divide a square into 2014 squares?

Third variation of the problem: You have paper cut-out squares with different length of sides. Try to build bigger squares using these smaller squares without overlapping and without gaps. Record the side length of the bigger squares you managed to build in your exercise book. Try to find a pattern in the number of smaller squares form which it is possible to build a bigger square. Is it possible to build a square whose sides are 2014 units long? Justify your general statement [4].

4. One Classroom Experience

In this section, first we give an outline of the background information related to the classroom experiment. For a more detailed description, please see [5] and [6].

4.1 The Action Research [7]

1) The teacher of the class was the researcher as well;

2) The school: a mixed comprehensive secondary school; students are 12-20 years old; selects the best achieving students from the area;

3) The students: sixteen 16-17 years old students took part; in the year of the experiment attended a class that specializes in mathematics and foreign languages; had a preparatory year with three mathematics lesson a week; in the next two years they had four mathematics lessons a week; in the year of the experiment the class followed the year 10 scheme of work for Hungarian secondary schools;

4) The lessons: 12 consecutive 45 minute-long lessons**;** 5 mathematical problems were discussed → 2-3 lessons for each problem; were planned using cooperative teaching techniques;

5) The problems: curriculum-based mathematics tasks; open-ended problems or investigations [8]; focused on developing different mathematical competencies [9];

6) Methods of data collection: “reflection booklets”—exercise books where students kept record of their problem solving activity; video recording half of the lessons; voice recording the group work; pre-, post and delayed mathematical tests; pre- and post-psychological questionnaires [10-11].

4.2 The Problem

For the first part of the experiment 5 problems were selected from different fields of mathematics (algebra, geometry, number theory, and combinatorics). Each of the problems was based on the Hungarian mathematics curriculum and they were either open problems already or there was the possibility for "opening" them [12]. For the complete list of the problems, please see [5].

The options for extension were suggested either by the teacher or by the students. For discussing these tasks twelve consecutive 45 minute long lessons were selected each of which were planned using cooperative teaching techniques [5]. The present article focuses on a problem field related to number tricks that can be explained using simple algebra. The heuristic strategies required to solve these problems are systematic thinking and thinking backwards [10].

Starter problem: Type the following mystic number in your calculator: 15873. Chose a number between 1 and 9 (including 1 and 9), then multiply 15873 by the chosen number. Multiply the result by 7. What do you notice? Try with more numbers. Can you explain what is going on [13]?

The problems [13]: Type your age in your calculator and multiply it by 1443 then multiply the result by 7. What do you notice? What can the explanation be?

Type 12 345 679 in your calculator and multiply it by 9, then by a positive one digit number, what do you notice? What can the explanation be?

Type an arbitrary three-digit number in your calculator then type the same digits again (so you can see a six-digit number of the form ABCABC). Divide this number by 13, and divide the result by 11. Finally, divide this result by 7. What do you notice? What can the explanation be?

For the next trick type 999 999 in your calculator then multiply it by a number between 1 and 6 (1 and 6 included). Divide the result by 7. What happens? Why?

More number magic (Time filler)!!!

Write the last 7 digits of your phone number on a piece of paper. Use these digits to form a seven-digit number that is different from your phone number. Subtract the smaller seven-digit number from the bigger one then divide the result by 9. Check the remainder. What do you notice? What can the explanation be?

4.3 Lesson Plan

1) The starter activity: The students were organized into groups of four [14] and they had to solve the starter activity using the cooperative structure *Think-Pair Share*. When applying this structure students are grouped at tables and a problem is presented to them. The team members are given a specific amount of time to think on their own about possible answers. Following this students discus their answers with each other [15];

2) The main tasks: The students worked in groups of four and we used cooperative techniques again. The structure applied this time is called *Pairs Check*. The steps for this method are:

a. Divide student into groups of four and have them work with their shoulder partner;

b. Give each team one worksheet with problems or questions (see problems above);

c. Partner *A* works the first problem or question while Partner *B* coaches and praises when necessary;

d. Partners reverse the roles;

e. After two problems are completed, pairs check with their partners across the table (face partners) (Team discussion);

f. Steps 3-5 are repeated for every two problems/questions [15].

4.4 Discussion

In this section, we present a student's solutions. It can be seen that mostly he followed the next pattern when trying to solve the problems: 1) systematic trial; 2) some kind of generalizations; 3) attempt to prove the statement.

4.4.1 Students' opinion

- "After trying more options everybody had an idea and we only had to choose the best one" (*KMM);*
- "It was easy because we could use a calculator and try some numbers" (*BZS);*

- “At first the problems looked like magic, but after thinking about them we saw that we had to use maths to solve them” (*KA);*
- “It was difficult that sometimes after many trials I still couldn’t see the solution” (*BT).*

4.4.2 Teacher’s comments

Based on the experience with the above presented problems and the other open problems used during the experiment we can say that there are several benefits of using open problems in teaching mathematical problem solving. First of all, open problems give the opportunity for the less able students to get started, since if nothing comes to their mind related to the solution they can start experimenting with concrete values or cases. Generalization and proving statements have always been a critical point for Hungarian students. When using open problems they can examine the different options which help them formulate a general statement and helps finding an idea for proof. The students taking part in the experiment were rather creative in terms of formulating general statements and giving a vague outline of the proof but they needed the teacher’s assistance in presenting their ideas mathematically. Because of the nature of open problems there might be more solutions or the same solution can be expressed in many different ways. Handling this in class is not always easy and is definitely time consuming. Using cooperative techniques seemed useful as working in groups provided opportunity for the students to be creative, to test their own ideas and to discuss different ways of solution. Taking everything into consideration using open problems contributes well to developing most students’ problem solving skills and even if it is time consuming the problems can be discussed in greater detail.

4.4.3 Future work

Formulating open problems—or problems which can be opened—that can be used in everyday teaching is one of the future tasks. Moreover, further experiments can be designed testing the effects of regular use of open problems on the development of the average students’ problem solving skills.

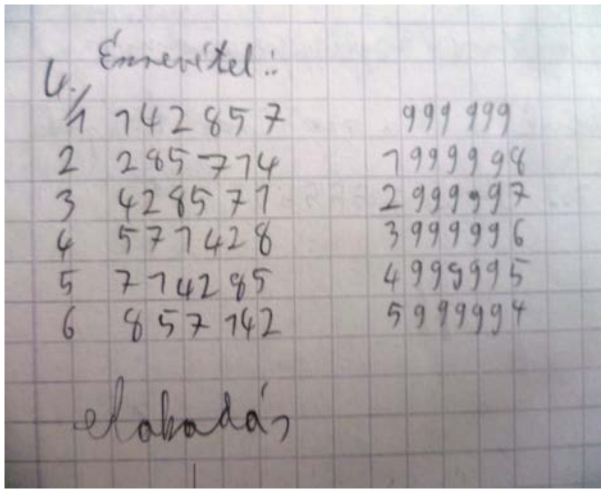

Fig.1 Systematic trial.

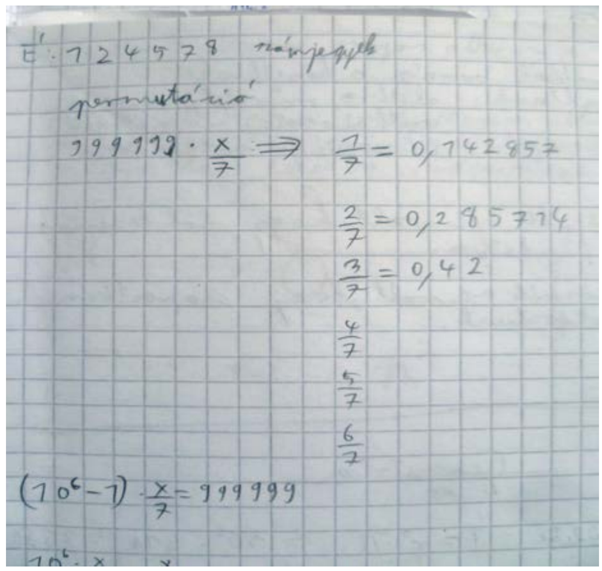

Fig.2 Attempt to prove.

5. Conclusions

The results of our investigations can be summarized in the following way.

1) The main point is the teacher—he/she needs to take the level of his (her) students into consideration [16];

2) Opening the problems helped the students start and continue the solution of the problems;

3) We need to accept the consequences of CLT, which states that there is a huge difference between

expert and novice students. Instead of using exclusively problem oriented instructions in case of new materials, problem-solving methods and guided instructions are more effective;

4) But how to spread these ideas? Three secondary school mathematics teachers are writing their PhD thesis in the following topics: cooperative methods, using guiding questions, using enactive and iconic representations in teaching mathematical problem-solving, using open problems. They present their results on local conferences and they write articles in educational journals;

5) An important task is to prepare future teachers to be more sensitive to learner—centred mathematics education with following the newest results of cognitive neuroscience.

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