Journal of Mathematical Sciences 2 (2015) 118-125 **Betty Jones & Sisters Publishing**

Solution of Strong Shocks Problem for Varying Density Medium

B.N. Prasad[1], Bulbul Sah[1], and R.K. Sharma[2]

1. Post-Graduate Department of Mathematics, Jamshedpur Women's College, Kolhan University, Jamshedpur- 831001, India

2. Department of Mathematics, B.A. College of Engineering & Technology, Jamshedpur, India

Abstract: The motion of strong shock as it propagates in the outer layer of a star far from its origin has been studied. The flow behind the shock is rendered approximately isothermal and so the spatial temperature gradient behind the shock is zero. The stage when shock traverses the outer layer, so that it is approximately planer, has been considered for gravitational effect to be negligible and the initial condition have been forgotten. An approximate analytic solution of this problem by employing Chernyii's expansion technique in which flow variables are expanded in series of powers of β , the density ratio across the strong shock has been obtained. The solution in closed form up to second order terms in β has been obtained. An analytic expression for α , the similarity exponent occurring in the law of shock propagation has been obtained from the consideration of singular point of the single differential equation, which is closed to the exact value. Solution so obtained using this technique is in excellent agreement with the exact solution of similar problem by previous researchers of the field. Finally approximate analytic solution of a strong shock approaching the surface of a star when the flow behind the shock is homothermal has been obtained.

Key words: Self-similar, similarity solution, homo-thermal flow, temperature gradient.

1. Introduction

In the study of rigid gas dynamics & shock propagation the self-similar processes are studied. It is of two types: self-similar motion of first kind & self-similar motion of second kind which has been studied by Zel'dovich and Raizer [1], Taylor [2], Sakurai [3], Sachdev & Ashraf [4], Ashraf & Prasad [5]. In self- similar solution of the first kind the similarity exponent α occurring in the law of shock propagation is determined from the dimensional considerations or laws of conservation themselves while in the solution of second kind this exponent α cannot be determined from these considerations in advance but is found from solving the differential equations which govern the flows.

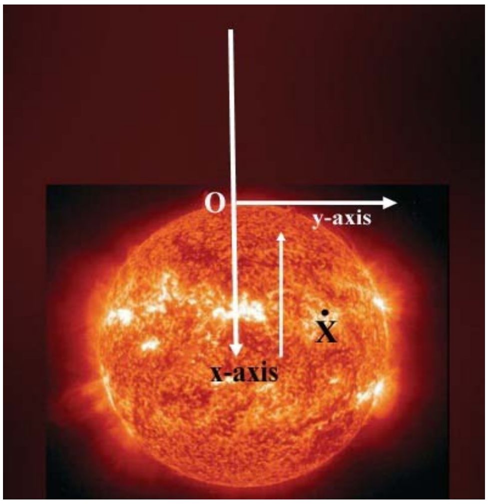
The solutions of self-similar motion of second kind are particularly interesting from the mathematical point of view since they are examples of solutions of differential equations which partially forget their initial conditions, Zel'dovich and Raizer [1], Meyer and Ho [6]. In this paper, we consider the motion of shock as it propagates in the outer layers of star far from its origin, near the centre. As the shock propagates in the outer layers of star, it accelerates and the temperature behind increases. Besides, the mean free path of radiation, which is inversely proportional to density, becomes very large so that there is intense transfer of energy by radiation, leading to the leveling down of temperature gradient. Thus the flow behind the shock is rendered approximately isothermal. The spatial temperature gradient is zero behind the shock. The time-dependent temperature goes on changing as the shock propagates and this temperature is different from that ahead of the shock. Sachdev and Ashraf [4] has solved this problem numerically.

In the present paper we have found an approximate analytic solution of this problem by employing Chernyii's [7] technique, in which the flow variables are expressed in series of powers of β, the density ratio

Corresponding author: B.N. Prasad, Ph.D., research fields: fluid mechanics. E-mail: bnprasad60@hotmail.com.

across the strong shock. We have found the solution in closed form up to second order terms in β. As remarked earlier, the similarity exponent α, occurring in the law of shock propagation, is obtained by solving an eigenvalue problem for a single differential equation to which the similarity equations are reducible. We have found an analytic expression for α, the similarity exponent, from the considerations of singular point of the single differential equation and this gives values of α closed to exact values. Comparison of our analytic solution with that of Sachdev and Ashraf [4], shows an excellent agreement. The error in the solution is $0(\beta^3)$ which is very small when γ is 0(1).

2. Basic Equations and Its Solutions



We take origin as the co-ordinates on the surface of the star and the positive x-axis points into the interior.

We assume that the undisturbed density ρ_0 ahead of the shock is given by

$$\rho_0 = bx^{\delta} \tag{2.1}$$

where b and δ are positive constants so that on the surface when $x = 0$ then $\rho_0 = 0$.

The time t is negative before the shock reaches the surface of the star and $t = 0$ is the instant at which the shock emerges at the surface. The shock position is assumed to be given by

$$X = A(-t)^{\alpha} \tag{2.2}$$

where A and α are constants.

The basic equations governing the one-dimensional isothermal flow behind the shock in terms of Lagrangian coordinate η and time t are

$$\frac{\partial x}{\partial \eta} = \frac{1}{\rho} \tag{2.3}$$

$$\frac{\partial^2 x}{\partial t^2} = -\frac{\partial p}{\partial \eta} \tag{2.4}$$

$$\frac{\partial T}{\partial x} = 0 \tag{2.5}$$

where x is the Eulerian co-ordinate and ρ , p and T are density, pressure and temperature behind the shock respectively .

Here η is the Lagrangian coordinate defined by

$$d\eta = \rho_0 dx_0 \tag{2.6}$$

where x_0 is the value of x at initial instant of time.

We introduce the similarity variable μ defined by

$$\mu = \frac{\eta}{\eta_s} \tag{2.7}$$

where η_s is the value of η at the shock.

Following Laumbach and Probstein [8] and Zel'dovich and Raizer [1], we assume that heat flux across the optically thin layer is continuous so that the classical shock condition hold. So for a strong shock, we have the boundary condition at the shock as

$$u_s = (1-\beta)\dot{X}$$
$$\rho_s = \frac{\rho_0}{\beta} \tag{2.8}$$
$$p_s = (1-\beta)\rho_0 \dot{X}^2$$

where ρ_0 is the undisturbed density, $\dot{X}$ is the shock velocity and β is the density ratio across a strong shock equal to $(\gamma-1)/(\gamma+1)$.

The equations (2.3), (2.4) and (2.5) may be also written as

$$\frac{\partial u}{\partial \eta} = -1/\rho^2 \frac{\partial \rho}{\partial t} \tag{2.9}$$

$$\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial \eta} \tag{2.10}$$

$$\frac{\partial T}{\partial x} = 0 \tag{2.11}$$

where u is the particle velocity behind the shock .

Following Laumbah and Probstein [8], we seek the solution in the similarity form as:

$$x = X\xi(\mu) \tag{2.12}$$

$$u = (1-\beta)\dot{X}v(\mu) \tag{2.13}$$

$$\rho = \frac{\rho_o(x)}{\beta} g(\mu) \tag{2.14}$$

$$p = (1-\beta)\rho_o(x)\dot{X}^2\pi(\mu) \tag{2.15}$$

where $v(\mu)$, $g(\mu)$, and $\pi(\mu)$ are reduced particle velocity, density and pressure respectively.

At the shock, where $\mu = 1$, boundary condition (2.8) for reduced function $v(\mu)$, $g(\mu)$, and $\pi(\mu)$ become

$$v(1) = g(1) = \pi(1) = 1 \tag{2.16}$$

The equation (2.11) and the gas law

$$p = R\rho T \tag{2.17}$$

together yield

$$\frac{d}{d\mu}\left\{\frac{\pi}{g}\right\} = 0 \tag{2.18}$$

Integrating equation (2.18) and using boundary condition (2.16) we get,

$$g(\mu) = \pi(\mu) \tag{2.19}$$

Substituting expression (2.12) to (2.15) in equations (2.3), (2.9) and (2.10) and making use of (2.19), we obtain

$$\frac{d\xi}{d\mu} = \frac{\beta}{1+\delta}\cdot\frac{1}{g} \tag{2.20}$$

$$\beta\delta + (1-\beta)(1+\delta)g\frac{dv}{d\mu} = \beta(1+\delta)\mu\frac{d\ln g}{d\mu} \tag{2.21}$$

$$\alpha^{-1}(1-\alpha)v + (1+\delta)\mu\frac{dv}{d\mu} = (1+\delta)g\frac{d\ln g}{d\mu} \tag{2.22}$$

Following Chernyii's [7], we expand the flow variables in series of powers of β, the density ratio across the strong shock,

$$\xi = \xi^{(0)} + \beta\xi^{(1)} + \beta^2\xi^{(2)} + \ldots \tag{2.23}$$

$$v = v^{(0)} + \beta v^{(1)} + \beta^2 v^{(2)} + \ldots \tag{2.24}$$

$$g = g^{(0)} + \beta g^{(1)} + \beta^2 g^{(2)} + \ldots \tag{2.25}$$

The boundary condition (2.16) may be expressed

$$\begin{aligned} &\xi^{(0)} = 1 \\ &\xi^{(1)} = \xi^{(2)} = \cdots = 0 \\ &v^{(0)} = 1 \\ &v^{(1)} = v^{(2)} = \cdots = 0 \\ &g^{(0)} = 1 \\ &g^{(1)} = g^{(2)} = \cdots = 0 \end{aligned} \tag{2.26}$$

We substitute expansion (2.23) to (2.25) into equations (2.20), (2.21) and (2.22) and obtain the following system of differential equations up to second order terms in β.

$$\frac{d\xi^{(0)}}{d\mu} = 0 \tag{2.27}$$

$$\frac{d\xi^{(1)}}{d\mu} = \frac{1}{(1+\delta)g^{(0)}} \tag{2.28}$$

$$\frac{d\xi^{(2)}}{d\mu} = -\frac{g^{(1)}}{(1+\delta)g^{2(0)}} \quad (2.29)$$

$$\frac{dv^{(0)}}{d\mu} = 0 \quad (2.30)$$

$$\frac{dv^{(1)}}{d\mu} = \frac{1}{g^{2(0)}}\left[\mu\frac{dg^{(0)}}{d\mu} - Bg^{(0)}\right] \quad (2.31)$$

$$\frac{dv^{(2)}}{d\mu} = \frac{1}{g^{2(0)}}\left[\mu\frac{dg^{(1)}}{d\mu} - \left\{2g^{(0)}g^{(1)} - g^{2(0)}\right\}\frac{dv^{(1)}}{d\mu} - Bg^{(1)}\right] \quad (2.32)$$

$$\frac{dg^{(0)}}{d\mu} = Cv^{(0)} \quad (2.33)$$

$$\frac{dg^{(1)}}{d\mu} = Cv^{(1)} + \mu\frac{dv^{(1)}}{d\mu} \quad (2.34)$$

$$\frac{dg^{(2)}}{d\mu} = Cv^{(2)} + \mu\frac{dv^{(2)}}{d\mu} \quad (2.35)$$

where $B = \delta/1+\delta$, $\lambda = (1-\alpha)/\alpha$, and $C = \lambda/1+\delta$.

We integrate equations(2.27) - (2.35) with boundary conditions (2.26) and finally obtain the solution as:

$$\xi^{(0)} = 1 \quad (2.36)$$

$$\xi^{(1)} = \frac{1}{(1+\delta)C}(\ln z - \ln K) \quad (2.37)$$

$$\xi^{(2)} = \ln z(\overline{D} + \overline{A}\ln z) + \frac{1}{z}(\overline{C} + \overline{B}\ln z) + \frac{\overline{E}}{z^2} + \overline{F} \quad (2.38)$$

$$v^{(0)} = 1 \quad (2.39)$$

$$v^{(1)} = A_1 \ln z + \frac{B_1}{z} + C_1 \quad (2.40)$$

$$v^{(2)} = \ln z(A_2\ln z + C_2) + \frac{\ln z}{z^2}(B_2\ln z + C_2) + \frac{1}{z^3}(D_2z^2 + E_2z + F_2) + H_2 \quad (2.41)$$

$$g^{(0)} = Cz \quad (2.42)$$

$$g^{(1)} = \ln z(D_1z + E_1) + F_1z - \frac{G_1}{z} + H_1 \quad (2.43)$$

$$g^{(2)} = \ln z\left[(A_3z + C_3)\ln z + (B_3z + D_3)\right] + \frac{\ln z}{z^2}\left[B_2z^2 + (G_2 - E_3)z - G_2k\right] + \frac{1}{z^3}\left[\begin{array}{l} D_2z^3 + (E_2 - G_3)z^2 \\ +(F_2 - H_3)z - F_2k \end{array}\right] + F_3z + I_3 \quad (2.44)$$

where $z = \mu + k$, $k = \frac{1-C}{C}$, $K = 1+k$

$$A_1 = \frac{1-B}{C}$$

$$B_1 = \frac{k}{C}$$

$$C_1 = -\left[A_1\ln K + \frac{B_1}{K}\right]$$

$$D_1 = CA_1$$

$$E_1 = (1-A_1)k - B_1$$

$$F_1 = CC_1 + A_1 - D_1$$

$$G_1 = B_1k$$

$$H_1 = -\left[(D_1K + K)\ln k + F_1K - \frac{G_1}{K}\right]$$

$$A_2 = -\frac{D^2{}_1}{2C^2}$$

$$B_2 = \frac{1}{C^2}[E_1(B + 2D_1) - D_1k(1 + B + 2D_1)]$$

$$C_2 = \frac{1}{C^2}[A_1 + C(C_1 + D_1) - F_1(B + 2D_1)]$$

$$D_2 = \frac{1}{C^2}\begin{bmatrix} B_2C^2 + B_1(1-C) \\ +k(CC_1 + 2A_1 - 2F_1D_1 - 2F_1 + C - F_1B) \\ +H_1(2D_1 + B) \end{bmatrix}$$

$$E_2 = \frac{1}{2C^2}\begin{bmatrix} k^2(1-A_1+D_1+2F_1) - k(E_1+2H_2) \\ -G_1(2+2D_1+B) \end{bmatrix}$$

$$F_2 = \frac{kG_1}{C^2}$$

$$G_2 = \frac{kD_1B_1}{C} - \frac{E_1B_1}{C}$$

$$H_2 = -\begin{bmatrix} \ln K(A_2\ln K + C_2)\dfrac{\ln K}{K^2}(G_2 + B_2K) \\ +\dfrac{1}{k^3}(K^2D_2 + KE_2 + F_2) \end{bmatrix}$$

$$A_3 = CA_2$$

$$B_3 = 2A_3(1-C) + CC_2$$

$$C_3 = \left(\frac{C-1}{2}\right)(2A_2k + B_2) - A_3k$$

$$D_3 = (C-1)[k(C_2 - 2A_2) + D_2] - B_3k$$

$$E_3 = (C-1)G_2 + B_2k$$

$$F_3 = H_2C - B_3 - C_2$$

$$G_3 = (C-1)(E_2+G_2) + D_2k$$

$$H_3 = \left(\frac{C-1}{2}\right)F_2 + E_2k$$

$$I_3 = -\begin{bmatrix} (A_3K + C_3)(\ln K)^2 + (B_3K + D_3)\ln K \\ +\dfrac{\ln K}{K^2}\times\left\{B_3K^2 + (G_2 - E_3)K - G_2k\right\} \\ +\dfrac{1}{k^3}\begin{Bmatrix} D_2K^3 + (E_2 - G_3)K^2 \\ +(F_2 - H_3)K - F_2k \end{Bmatrix} + F_3K \end{bmatrix}$$

$$\overline{A} = -\frac{D_1}{2(1+\delta)C^2}$$

$$\overline{B} = \frac{E_1 - kD_1}{(1+\delta)C^2}$$

$$\overline{C} = \overline{B} + \frac{H_1 - kF_1}{(1+\delta)C^2}$$

$$\overline{D} = -\frac{F_1}{(1+\delta)C^2}$$

$$\overline{E} = -\frac{G_1}{2(1+\delta)C^2}$$

$$\overline{F} = -\begin{bmatrix} \ln K(\overline{A}\ln K + \overline{D}) + \dfrac{\overline{B}\ln K}{K} \\ +\dfrac{1}{K^2}(K\overline{C} + \overline{E}) \end{bmatrix}$$

From equations (2.21) and (2.22) we obtain

$$\frac{dv}{d\mu} = \frac{\beta\left[\alpha^{-1}(1-\alpha)\mu v - g\delta\right]}{(1+\delta)\left[(1-\beta)g^2 - \beta\mu^2\right]} \qquad (2.45)$$

As pointed out earlier, for the second kind of self-similar motions, the value of similarity exponent α, occurring in the law of shock propagation is not provided by the physical considerations but is obtained by solving the single ordinary differential equation (2.45) which possesses several singular points. The solution is single valued and physically meaningful when $dv/d\mu$ is finite in the flow field and for this the numerator and the denominator in the equations (2.45) must vanish simultaneously. The numerator and the denominator vanish along the path,

$$v = 1, g = \left(\frac{\beta}{1+\beta}\right)^{1/2}\cdot\mu \qquad (2.46)$$

This gives an analytic expression for α, the similarity exponent, as

$$\alpha = \left[1 + \delta\left(\frac{\beta}{1-\beta}\right)^{1/2}\right]^{-1} \qquad (2.47)$$

3. Results and Discussions

We have found an approximate analytic solution, given by equations (2.36) to (2.44), of a strong shock approaching the surface of a star when the flow behind the shock is homo-thermal. The solution has been obtained in a closed form up to second order terms in β, the density ratio across the strong shock. To the zeroth order approximation the particle velocity behind the shock is everywhere the same as the shock velocity while the density and pressure are linear functions of μ. The first order terms contribute much more to velocity than to the density. C and K are positive and thus the reduced density and pressure increases behind the shock as μ increases. The Eulerian distance x, to the zeroth order approximation is the same as shock distance and hence the contribution from the first order term is significant. Equations (2.36) to (2.38) express the Eulerian similarity variable ξ in terms of the Lagrangian similarity variable μ. The first and second order terms in the solution are too complicated to lend themselves to visual estimation.

TABLE – I
Values of α obtained from analytic expression.

γ \ δ	3.25	2	1	0.5
5/3	0.3476	0.4641	0.6339	0.7760
7/5	0.4075	0.5278	0.6909	0.8172
6/5	0.4931	0.6125	0.7597	0.8634

TABLE – II
Values of α obtained from numerical integration.

γ \ δ	3.25	2	1	0.5
5/3	0.3136	0.4346	0.6146	0.7670
7/5	0.3690	0.4949	0.6720	0.8090
6/5	0.4568	0.5835	0.7442	0.8572

TABLE – III
Values of α obtained from.
(Sakurai[3] & Zel'dovich & Raizer[1])

γ \ δ	3.25	2	1	0.5
5/3	0.590	0.6966	0.8174	0.8976
7/5		0.7177	0.8318	0.9062
6/5		0.7514	0.8544	0.9195

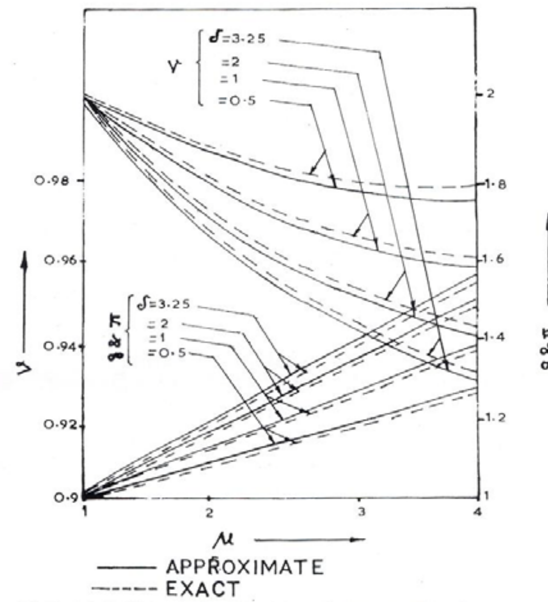


FIG. 1. FLOW VARIABLE DISTRIBUTIONS BEHIND THE SHOCK WAVE FOR γ=6/5

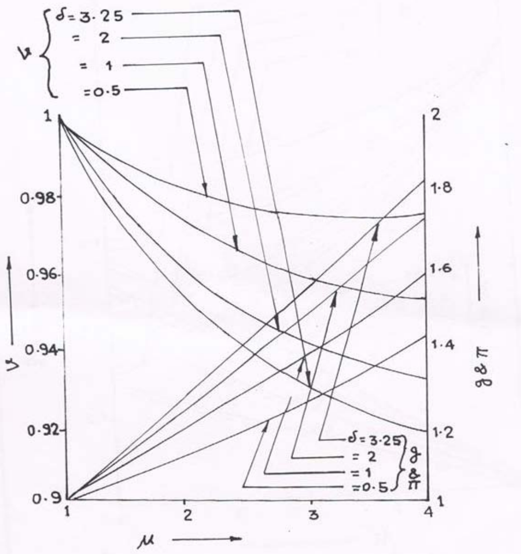


FIG. 2 FLOW VARIABLE DISTRIBUTIONS BEHIND THE SHOCK WAVE FOR γ= 7/5

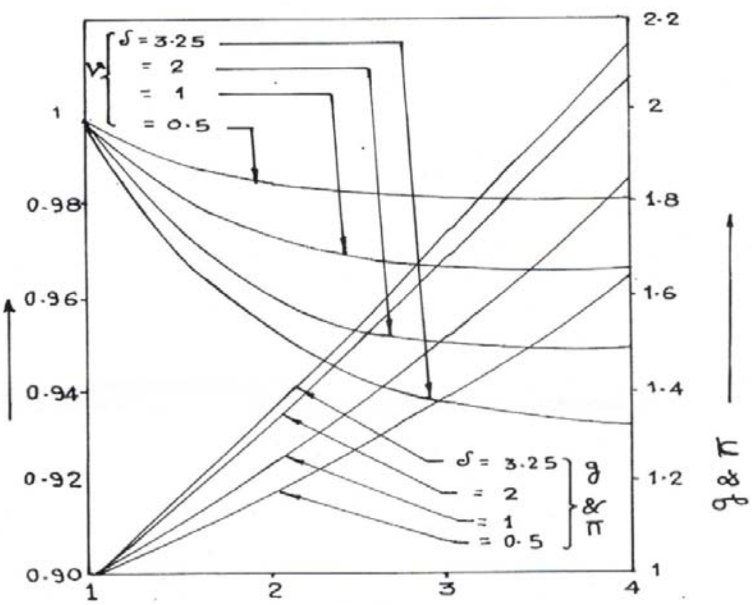


FIG.3 FLOW VARIABLE DISTRIBUTIONS BEHIND THE SHOCK WAVE FOR γ= 5/3

Table I gives the values of the similarity exponent α for different values of $\delta = 3.25,\ 2,\ 1,\ 0.5$ the density exponent and $\gamma = 5/3, 7/5$ *and* $6/5$, the ratio of specific heats, as calculated from expression (2.47).

The value of α for $\gamma = 5/3,\ and\ \delta = 3.25$ is particularly important in the context of stellar envelopes in radiative equilibrium. Table II gives the values of α obtained in numerical integration. The difference between the values of α , obtained from the equation (2.47) and the numerical integration , reduces considerably for all γ when δ decreases. Table III gives the value of α for the adiabatic flow, studied by Sakurai [3], Zel'dovich and Raizer [1], Nadezhin and Franks- Kamenetskii [9]. We find from table I and table III that the value of α for the isothermal case, for all γ and δ that we have considered , is less than for the adiabatic case ,showing that the shock velocity $\dot{X}$ $\propto X^{-(1-a)/a}$ in the isothermal case , becomes much greater near the surface of the star compared to that in the adiabatic case . The reduced functions g and π are not bounded at $\mu = \infty$ for which corresponds to either $t = 0, x \neq 0$ or the point far behind the shock for $t < 0$.

Thus in the isothermal flow it is not possible to consider the flow behind the shock when it reaches the surface of the star or the flow thereafter. In fact the self-similar flow is valid in a small region near the surface and our solution gives fairly good results in the limited region.

Figure I depicts the distribution of reduced functions v , g and π showing the velocity, the density and pressure behind the shock respectively at any time for $\gamma = 6/5$. The corresponding exact values are shown by dotted curves. Our analytic solution is in good agreement with the exact solutions. Sachdev and Ashraf [4] have compared the isothermal flow with the adiabatic flow. Figure II & III show the distribution of v , g and π for $\gamma = 7/5$ and 5/3 respectively for the isothermal flow only.

References

[1] Y.B. Zel'dovich, Y.P. Raizer, Physics of Shock Waves and High Temperature Hydrodynamic Phenomena, Dover Publications, 2002.

[2] G.I. Taylor, The formation of blast wave by a very intense explosion, Proc. Roy. Soc. 201 (1950) 159-174.

[3] A. Sakurai, On the problem of a shock wave arriving at the edge of gas, Commun. Pure and Appl. Maths., 13 (1960) 353.

[4] P.L Sachdev, S. Ashraf, Strong shock with radiation near the surface of a star, Phys. of Fluids 14 (10) (1971) 2107-2110.

[5] S. Ashraf, B.N. Prasad, Analytic solution of self-similar strong shock in an exponential medium for zero temperature gradient, ZAMP Zeitschrift für angewandte Mathematik und Physik 39 (2) (1988) 233-241.

[6] R.E. Meyer, D.V. Ho, Notes on non uniform shock propagation,Jour. Accoust. Soc. Amr. 35 (1963) 1126.

[7] G.G. Chernyi, The problem of a point Explosion, Dokl.Akod.Nauk, SSSR 112 (1957) 213-216;

[8] D.D. Laumbach, R.P. Probstein, Self-similar strong shocks with radiation in decreasing exponential atmosphere, Phys. Fluids 13 (1970) 1178-1183.

[9] D.K. Nadezhin, D.A. Frank-Kamanetskii, Particle acceleration in a shock emerging from a star, Sovt. Astro. 8 (1964) 655-674