Journal of Mathematical Sciences 2 (2015) 155-170
ISSN 2372-5214

Betty Jones & Sisters Publishing
http://www.bettyjonespub.com

Bochner Integral and Differentiation for Vector-Valued Functions in Arbitrary Locally Convex Topological Vector Space

I. Hadi, A. Mbarki, S. Lahrech, A. Ouahab, A. Benbrik

Department of Mathematics and Computer Sciences, Faculty of Sciences, Mohamed First University, Oujda, Morocco

Abstract: In this work, first we will define another measurability notion with respect to the topology of semi-normed spaces which is weaker than the usual strong meausurability. Next, we introduce some Bochner-Lebesgue and Bochner-Sobolev spaces more generalized to extract all fundamental properties known in integration theory and that are necessary notions in the study of abstract differential equations when solutions take values in a locally topological vector space.

Keywords: Locally convex topological vector space, bi-topological space, measure theory, Bochner space, Lebesgue space, Sobolev space, embedding operator.

1. Introduction

In the study of strong measurability for Banach space-valued functions, there are many cases in which this kind of measurability appears heavy or impossible for a considerable collection of functions. As an example to illustrate this phenomenon:

Let $X = L^1_{loc}(0,1)$ where (0, 1) is an open interval in IR. Then X is a locally convex topological vector space equipped by the separating family of semi-norms:

$$P_K(f) = \int_K |f(x)| dx,$$

Consider the vector-valued function defined as follow:

$$f : (0,1) \mapsto L^1_{loc}(0,1)$$

$$t \mapsto f_t(x) = \frac{t}{x}.$$

So, the vector-valued function which belongs in the above example is continuous with respect to the topology generated by the family of semi-norms $(P_K)_K$. Hence, the introduction of spaces denoted $L^p((0,1),(X,\tau))$ whose will be our main in this note instead usual Bochner-spaces $L^p((0,1),(X,\|.\|))$.

In parallel, we will study other class of spaces denoted $L^p((0,1),(X_\tau,\|.\|))$ of functions which are not strongly measurable but measurable with respect to its topology τ and their norms have a finite Lebesgue integral, for example:

Let H a Hilbert space whose dimension is equal to the cardinality of IR. Let $\{e_t; t \in (0,1)\}$ be an orthonormal basis of H equipped by its norm and its weak topology seen as a locally convex topology compatible with its vector structure and define a function:

$$f : (0,1) \mapsto H$$

$$t \mapsto f(t) = e_t$$

Recall that a vector-valued function is strongly measurable if it is a limit of simple functions with respect to the norm. This property is tested usually by the Pettis theorem (1938), see [1], which asserted that a strongly measurable function f should be almost

Corresponding author: I. Hadi, Ph.D., professor, research fields: functional analysis and differential equations. E-mail: i.hadi@gmx.fr.

separably valued, then f is weakly but not strongly measurable and $\int_0^1 \| f(t) \| dt = 1 < \infty$.

Furthermore, a Riemann integrable function $f : ((0,1) \mapsto (X, \|.\|)$ is not necessarily norm continuous almost everywhere with respect to the Lebesgue measure either weakly continuous. There are a large literature in the study of the relationship between Riemann integrability and continuity of functions, we refer to the works of Alexiewiez and Orlicz [2] and Kadets [3], extended by Chonghu Wang and Zhenhua Yang [4] to arbitrary locally convex topologies on X weaker than the norm topology.

The purpose of this note is the study of vector-valued functions in an arbitrary locally convex topological vector space (X, τ) which are measurable with respect to the topology τ, meaning, functions whose are continuous with respect to the topology τ on each $K_n \in (0,1)$ of any compacts sequence $(K_n)_n$ such that $\mu(K_n) \to 1$ as $n \longrightarrow \infty$, where μ is the Lebesgue measure in IR. So, basing on this notion of measurability we will define a class of Bochner integrable functions for studding their differentiabilities to introduce Bochner-Sobolev spaces associated to those class of functions. Hence, resolving abstract differential equations with less regularity.

2. Preliminaries and Definitions

We throughout assume that (X, τ) is a locally convex topological vector space and if the space X is equipped also with a norm, then X becomes a bi-topological vector space under the pair of topologies (τ, ||.||).

Definition 2.1
We say that (X, τ) is a locally convex topological vector space when :

$1 - \tau$ is compatible with the vector structure of X i.e: The two following mappings are continuous:

$X \times X \longrightarrow X$
$(x, y) \longrightarrow x+y$

and

$\mathrm{IR} \times X \longrightarrow X$
$(\lambda, x) \longrightarrow \lambda x.$

$2 - \tau$ *is locally convex i.e:*

$\forall_x \in X$, $\forall U \in \tau$, $\exists V \in \tau$ convex such that $x \in V \subset U$.

Recall that by the theorem of Hyers [5], any locally convex topological vector space could be equipped by a family of semi-norms which generates its topology. We refer also to the work of P. K. Kamtan [6].

Definition 2.2
We say that $q : X \mapsto IR^+$ is a semi-norm if:

(i) $q(\lambda x) = |\lambda|\, q(x)$ $\forall x \in X, \lambda \in IR$

(ii) $q(x+y) \le q(x) + q(y)$ $\forall x, y \in X$

when a family of semi-norms Q is separating we have :

(iii) $(\forall_q \in Q\; q(x) = 0) \Longrightarrow x = 0$.

Remark 2.1 In particular, a norm is a separating semi-norm.

Example 2.1 - Let $X := C(\Omega)$, Ω an open of IR^N, space of all continuous functions functions on Ω. Put for any $K \subset \Omega$ compact

$$(P_K)(\varphi) = \sup_{x \in K} |\varphi(x)|,$$

so $(P_K)_K$ is a separating family of semi-norms on X.

Example 2.2 - Let $X := L^1_{loc}(\Omega)$, Ω an open of IR^N, $p \in [1, \infty)$, a vector space of measurable functions $f : \Omega \to$ IR such that $\forall K \subset \Omega$ compact we have $f \in L^p(K)$. Then the family $P_K(f) = \| f \| L^p(K)$ is a separating semi-norms on X.

Remark 2.2 The weak topology of a normed vector space could be seen as a locally convex topology generated by the separating family of semi-norms

$P_f(x) = |f(x)|$ where $f \in X'$ and X' is the dual of X.

Definition 2.3

Let Q a separating family of semi-norms on X. We say that a sequence $(x_n)_n$ of X converges to x with respect to Q, we denote $x_n \to x$ with respect to Q if $\forall q \in Q, q(x_n - x) \to 0 \text{ when } n \to \infty$.

Definition 2.4

Let (X, τ, Q) and (Y, ζ, P) two locally convex topological vector spaces. We say that a function $\varphi : X \mapsto Y$ is continuous if and only if $\forall_\varepsilon > 0, \forall_x \in X, \forall_p \in P, \exists \eta > 0, J$ finite set such that: $\forall_j \in J \forall_y$ *such that* $q_j(y - x) < \eta \Rightarrow p(\kappa(x) - \kappa(y)) < \varepsilon$.

Definition 2.5

Let (X, τ) a locally convex topological vector space. We say that $B \subset X$ is bounded if for each $q \in Q \exists Mq > 0$ such that

$$\sup_{x \in B} q(x) \leq Mq.$$

Definition 2.6

We say that (X, τ) is sequentially complete, if every Cauchy sequence in (X, τ) converges in (X, τ).

3. Bochner Integral for Uniformly Continuous Vector-Valued Functions

Theorem 3.1

Let (X, τ) a sequentially complete space, $0 < T < \infty$, consider an uniformly continuous function $f : (0, T) \mapsto (X, \tau)$ and $I_1, \ldots I_n$ is a finite collection of non-overlapping closed sub-intervals covering $I = (0, T)$ such that

$\delta n = \max_{1 \leq k \leq n} \mu(I_k)$, t_k any element of I_k. Then, there exists

$$\int_0^T f(t)dt = \lim_{\delta n \to 0} \sum_{k=1}^{n} f(t_k)\mu(I_k)$$

where the limit is taken with respect to the topology τ.

Proof.

For each subdivision with n subintervals we put

$$S_n = \sum_{k=1}^{n} f(t_k)\mu(I_k)$$

to claim the result, it suffices to show that $(S_n)_n$ is a Cauchy sequence in (X, τ) which is sequentially complete. Let:

$$S_n = \sum_{k=1}^{n} f(t_k)\mu(I_k)$$

and

$$S_m = \sum_{k=1}^{m} f(t'_k)\mu(I'_k)$$

Take S_{m+n} as an union of the both subdivisions S_n and S_m so we have:

$$S_{m+n} = \sum_{k=1}^{m+n-1} f(t'')\mu(I''_k),$$

without loss of generality we take $(t_j := t_k, 1 \leq j \leq m+n-1)$ since $I''_j \subset I_k$ and $(t'_j := t'_k, 1 \leq j \leq m+n-1)$ if $I''_k \subset I'_k$. Then for any semi-norm $q \in Q$ we get:

$$q(S_m - S_n) \leq q(S_n - S_{m+n}) + q(S_{m+n} - S_n)$$
$$\leq \sum_{k=1}^{m+n-1} q(f(t''_k) - f(t_k))\mu(I''_k) +$$
$$\sum_{k=1}^{m+n-1} q(f(t''_k) - f(t'_k))\mu(I''_k)$$

Since f is uniformly continuous, so, $\forall_\varepsilon > 0$,

$\forall_q \in Q, \exists_\eta > 0$, such that

$$\forall_t, t' \in (0, T), |t - t'| < \eta \Rightarrow q(f(t) - f(t')) < \frac{\varepsilon}{2}$$

and $\exists_{n_0}$ such that $\forall_n \geq n_0, \delta_n < \eta$. Thus, for $\forall_{m,n} \geq n_0$ we get $q(S_m - S_n) < \varepsilon\mu(I)$.

Which proves that $(S_n)_n$ is a Cauchy sequence in (X,τ), then it converges in (X,τ) to an unique limit since Q is a separating family of semi-norms on X and this limit do not depend on the choice of subdivisions, to show that we proceed by the same as above.

Proposition 3.1

Let $f : (0, T) \mapsto (X, \tau)$ an uniformly continuous. Then for any semi-norm $q \in Q$ we have

$$q(\int_0^T f(t)dt) \leq \int_0^T q(f(t))dt.$$

Proof.

Using the definition of Bochner integral in theorem (3.1), the definition of Lebesgue integral and since the semi-norm $q \in Q$ is lower semi-continuous we get

$$q(\int_0^T f(t)dt) \leq \liminf_{\delta_n \to 0} q(\sum_{k=1}^n f(t_k)\mu(I_k))$$

$$\leq \liminf_{\delta_n \to 0} \sum_{k=1}^n q(f(t_k))\mu(I_k) \leq \int_0^T q(f(t)).$$

Thus, we achieve the proof.

Proposition 3.2

Let $x' \in (X,\tau)'$ the dual of (X,τ) . Then there exists a constant $C > 0$ and a finite family of semi-norms $(q_j)_{j\in J} \in Q$ such that

$$x'(\int_0^T f(t)dt) = \int_0^T x'(f(t))dt,$$

$$|x'(\int_0^T f(t)dt)| \leq C\int_0^T \sup_{j\in J} q_j(f(t))dt.$$

Proof.

It is an immediately result of the theorem (3.1), the proposition (3.1) and the fact that for any continuous mapping $x' \in (X,\tau), \exists C > 0$ such that

$$\forall_x \in X, |<x', x>| \leq C \sup_{j\in J} q_j(x).$$

4. Measurability with respect to the Topology τ and Generalized Bochner Integral

Definition 4.1

Let a function. $f : (0,T) \mapsto (X,\tau)$ We say that f is measurable with respect to the topology τ if $\forall_\varepsilon > 0, \exists K \subset I = (0,T)$ a compact such that $\mu(I \setminus K) < \varepsilon$ and $f \mid K : K \mapsto (X,\tau)$ is continuous.

Proposition 4.1

Let $f : (0,T) \mapsto (X,\tau)$ a measurable functions with respect to τ and $q \in Q$ any lower semi continuous semi-norm. Then $qof : (0,T) \mapsto IR$ *is a Lebesgue measurable function.*

Proof. Suppose that f is measurable with respect to τ. Then $\forall_\varepsilon > 0, \exists K \subset I = (0,T)$ a compact such that $\mu(I \setminus K) < \varepsilon$ and $f \mid K : K \mapsto (X,\tau)$ is continuous, in the other hand $qof : (K \mapsto IR$ is lower semi continuous, hence Lebesgue measurable. By the Luzin theorem, there exists $K_1 \in K$ a compact such that $\mu(K \setminus K_1) < \frac{\varepsilon}{2}$ and $qof : (K_1 \mapsto IR$ is continuous, then $\mu(I \setminus K_1) < \varepsilon$, so, again by Luzin theorem we deduce that qof is measurable.

Remark 4.1

If $(X,\tau,\|.\|)$ is a bi-topological such that the norm function $\|.\|: (X,\tau) \mapsto IR^+$ is lower semi continuous, then, for every measurable function $f : ((0,T) \mapsto (X,\tau))$ with respect to τ the function $\| f \|: (0,T) \mapsto IR^+$ is Lebesgue measurable.

Proposition 4.2

Let $f:(0,T)\mapsto(X,\tau)$ a measurable functions with respect to τ such that $qof \in L^1(0,T)$ for every semi-norm $q\in Q$. Then

$$\int_0^T f(t)dt=\lim_{n\to\infty}\int_{K_n} f(t)dt$$

for any sequence $(K_n)_n$ of compacts such that $\mu(I\setminus K_n)\to 0$ and $f\mid K_n$ is continuous. This limit does not depend on the choice of compacts sequence.

Proof.

Since f is measurable with respect to τ; then there exists $(K_n)_n$ a sequence of compacts in (0, T) such that $\mu(I\setminus K_n)\to 0$ *and* $f\mid K_n$ is continuous, so it is uniformly continuous with respect to τ on K_n. Hence, $\int_{K_n} f(t)dt$ is well defined (see theorem (3.1)). To claim the result we need to show that $(\int_{K_n} f(t)dt)_n$ is a Cauchy sequence in (X, τ). Indeed, Let q a semi-norm, then

$$\begin{aligned}
&q(\int_{K_m} f(t)dt-\int_{K_n} f(t)dt)\\
&\le q(\int_{K_m\setminus K_n} f(t)dt)+q(\int_{K_n\setminus K_m} f(t)dt)\\
&\le \int_{K_m\setminus K_n} q(f(t))dt+\int_{K_n\setminus K_m} q(f(t))dt\\
&\le \int_{I\setminus K_n} q(f(t))dt+(\int_{I\setminus K_m} q(f(t))dt.
\end{aligned}$$

Since, $qof \in L^1(0,T)$, $\mu(I\setminus K_n)\to 0$ and the Lebesgue integral is absolutely continuous. Therefore, $(\int_{K_n} f(t)dt)_n$ is a Cauchy sequence in (X, τ). Consequently, it converges since (X, τ) is sequentially complete. Similarly, we prove that the limit $\int_0^T f(t)dt$ do not depend on the choice of the compacts sequence $(K_n)_n$. Thus, we get the result.

Remark 4.2

The propositions (3.1), (3.2) could be generalized to any f measurable function with respect to τ such that $qof \in L^1(0,T)$ for every semi-norm $q\in Q$, by using the dominated convergence theorem of Lebesgue.

Now, we are ready to define the following Bochner spaces:

$L^p((0,T),(X,\tau)):=\{f:(0,T)\mapsto(X,\tau)\mid f$ is measurable with respect to τ and $\int_0^T (qof)^p dt<\infty, \forall_q \in Q\}$.

If we denote by $(A_q^p)q\in Q$ a family such that for any $f\in L^p((0,T),(X,\tau))$ and $q\in Q$ we have

$$(A_q^p)(f):=(\int_0^T (qof)^p dt)^{\frac{1}{p}}.$$

We can see that $(A_q^p)q\in Q$ is a separating family of semi-norms on $L^p((0,T),(X,\tau))$, then it becomes a locally convex topological vector space under the topology generated by those semi-norms.

Analogously, we define for every $q\in Q$

$$A_q^\infty(f):=ess\sup_{t\in(0,T)} q(f(t)).$$

and

$L^\infty((0,T),(X,\tau)):=\{f:(0,T)\mapsto(X,\tau)\mid f$ is measurable with respect to τ and $A_q^\infty(f)<\infty, \forall_q \in Q\}$.

A question that we can pose is: if we can define the usual Lebesgue-Bochner spaces when $(X,\tau,\|.\|)$ is a bi-topological space but vector-valued functions there should be taken measurable with respect to the topology τ?

The answer could be asserted under some assumptions on the relationship between the two topologies of $(X,\tau,\|.\|)$. Denote:

B_1: the unit ball of $(X,\|.\|)$.

Lemma 4.1

Let $(X,\tau,\|.\|)$ is a bi-topological vector space such that τ is locally convex topology and $\|.\|$ is a norm on X. Suppose that B_1 is bounded on (X,τ). Then $\forall_x \in X, \forall_q \in Q, \exists C_q > 0$ such that

$$q(x) \le C_q \|x\|$$

Proof.

For $x \in X, x = 0$ the result is true. Let suppose $x \in X, x \neq 0$ then $\frac{x}{\|x\|} \in B_1$, which is bounded in (X, τ). Hence, for any $q \in Q$ a semi-norm on X, there exists $C_q > 0$ such that

$q(\frac{x}{\|x\|}) \le C_q$, which claim the result.

Remark 4.3

If B_1 is bounded in (X,τ), then every Cauchy sequence in $(X,\|.\|)$ is a Cauchy sequence in (X,τ).

Lemma 4.2

Let $(X,\tau,\|.\|)$ is a bi-topological vector space and suppose that B_1 is closed in (X,τ). Then the norm function $\|.\|: (X,\tau) \mapsto IR^+$ is lower semi-continuous.

Proof.

Set $S_\lambda = \{x \in X; \|x\| \le \lambda\}$.

To claim the result we should show that S_λ is closed in *(X, τ)* and the topology τ is compatible with the vector structure of X, S_λ is also closed in *(X, τ)*, therefore the result is deduced.

Remark 4.4

Recall that in the case of a reflexive normed space X, the unit Ball is weakly compact which is a strongly condition than our two assumptions here.

By the remark (4.1), we can define the following Bochner spaces when $(X,\tau,\|.\|)$ is a bi-topological vector space:

$L^p((0,T),(X,\|.\|)) := \{ f : (0,T) \mapsto (X,\tau) \mid f$ is measurable with respect to τ and $\int_0^T \|f\|^p\, dt < \infty\}$.

Put

$$|f\|_p = (\int_0^T \|f\|^p\, dt)^{\frac{1}{p}}.$$

Then $L^p((0,T),(X,\|.\|))$ is a normed space under the norm $\|.\|_p$.

Put

$$\|f\|_\infty = ess \sup_{t\in(0,T)} \|f(t)\|.$$

Analogously, we define

$L^\infty((0,T),(X_\tau,\|.\|)) := \{ f : (0,T) \mapsto (X,\tau) \mid f$ is measurable with respect to τ and $\|f\|_\infty < \infty\}$.

Theorem 4.1

If the unit ball B_1 is bounded and closed in (X, τ), then $L^1((0,T),(X_\tau,\|.\|))$ is Banach space under the norm $\|.\|_1$.

Proof.

Let (f_n) a Cauchy sequence in $L^1((0,T),(X_\tau,\|.\|))$. Thus, there exists a subsequence $(f_{n_k})_n$ such that for $n \ge n_k$ we have $\|f_n - f_{n_k}\|_1 < 2^{-k}$.

Pick any ε > 0. As (f_n) is a measurable function with respect to τ; then $\forall_n \in IN \exists K_n \subset I$ a compacts such that $\mu(I \setminus K_n) < \varepsilon 2^{-n-1}$ and $f \mid K_n : K_n \mapsto (X,\tau)$ is continuous. Put $K' = \bigcap_{n=1}^{\infty} K_n$.

Then $\mu(I \setminus K') \le \frac{\varepsilon}{2}$. Consider

$$\varphi_n(t) = \sum_{k=1}^{n} \| f_{n_{k+1}}(t) - f_{n_k}(t) \|$$

So,

$$\|\varphi_n\|_1 \le \sum_{k=1}^{n} \| f_{n_{k+1}}(t) - f_{n_k}(t) \|_1 = \sum_{k=1}^{n} 2^{-k} < 1$$

and $(\varphi_n)_n$ is an increasing sequence of Lebesgue integrable functions, so by using Bepo-Levi theorem φ_n converges a.e. on (0, *T*) to a Lebesgue integrable function φ. In the other hand, for $m \ge n \ge 2$ and without loss of generality we denote f_k instead f_{n_k} we get:

$\| f_m(t) - f_n(t) \| \leq \varphi(t) - \varphi_{n-1}(t)$.

Hence, $(f_n)_n$ is a Cauchy sequence in $(X, \|.\|)$ and by the remark (4.3) it is also a Cauchy sequence in (X, τ), so it converges to certain $f \in (X, \tau)$. Let show that f is a measurable function with respect to τ, for that we need the locally uniform convergence in (X, τ) of the sequence $(f_n)_n$ to f. Indeed, recall that the sequence $(\varphi_n)_n$ converges a.e. on $(0, T)$ to φ, so by Egorov theorem $\exists K'' \subset I$ compact such that $\mu(I \setminus K'') < \frac{\varepsilon}{2}$ and $(\varphi_n)_n$ converges uniformly to φ on K''. Then by the lemma (4.1), for any semi-norm $q \in Q$ and $t \in K''$ we have $\forall_{m,n} \geq 2$

$$q(f_m(t) - f_n(t)) \leq C_q \| f_m(t) - f_n(t) \|$$
$$\leq C_q \sup_{t \in K''} \| \varphi(t) - \varphi_{n-1}(t) \|.$$

Hence, $(f_n)_n$ converges uniformly on K'' in (X, τ) to the function f and if we take $K = K' \bigcap K''$, we deduce that f_n is continuous on K with respect to the topology τ for any $n \geq 2$ and $\mu(I \setminus K) < \varepsilon$. Consequently, f is measurable with respect to τ. Furthermore, by the lemma (4.2) the norm $\|.\|$ is lower semi continuous on (X, τ) and since $f_n \to f$ in (X, τ) then

$$\| f \| \leq \liminf_{n \to \infty} \| f_n \|,$$

So, since f_n is a Cauchy sequence by the norm, then it is uniformly bounded and by Fatou lemma we get

$f \in L^1((0,T), (X_\tau, \|.\|))$.

Also, by the fact that $\forall_m \geq n \geq 2$

$\| f_m(t) - f_n(t) \| \leq \varphi(t) - \varphi_{n-1}(t)$,

$f_m \to f$ in (X, τ) and $\|.\|$ is lower semi continuous on (X, τ). Then $\forall_n \geq 2$

$$\| f(t) - f_n(t) \| \leq \liminf_{m \to \infty} \| f_m(t) - f_n(t) \| \leq \varphi(t) - \varphi_{n-1}(t).$$

Hence, (f_n) converges to f in $(X, \|.\|)$. In the other hand, $\sup_m \| f_m - f_n \|_1 \leq \| \varphi \|_1 < \infty$

So, we deduce by Fatou lemma

$\| f(t) - f_n(t) \|_1 \leq \liminf_{m \to \infty} \| f_m(t) - f_n(t) \|_1 \leq 2^{-n}$.

Therefore, $\| f(t) - f_n(t) \|_1 \to 0$ i.e., $f_n \to f$ in $L^1((0,T), (X_\tau, \|.\|))$.

Corollary 4.1

Let $f, f_n \in L^1((0,T), (X_\tau, \|.\|))$ and $f_n \to f$ in $L^1((0,T), (X_\tau, \|.\|))$. Then, there exists a sub-sequence (f_{n_k}) in $L^1((0,T), (X_\tau, \|.\|))$ which converges a.e. on $(0, T)$ to f with respect to the norm.

Proof.

Since $f_n \to f$ in $L^1((0,T), (X_\tau, \|.\|))$, then $(f_n)_n$ is a Cauchy sequence in $L^1((0,T), (X_\tau, \|.\|))$, so we proceed analogously as the previous argument of the theorem (4.1) we deduce that there exists a sub-sequence denoted f_{n_k} which converges to f^* in $(X, \|.\|)$ and in $L^1((0,T), (X_\tau, \|.\|))$. Hence, by the uniqueness of the limit in $L^1((0,T), (X_\tau, \|.\|))$ we conclude that $f \equiv f^*$ *a.e* on $(0, T)$.

Remark 4.5

By a similar proof as in theorem (4.1) we show that for every $p \in [0, \infty)$ the space $L^p((0,T), (X_\tau, \|.\|))$ is a Banach space under the norm $\|.\|_p$.

Theorem 4.2

(Lebesgue-Bochner Dominated Convergence Theorem)

Suppose that $f_n \in L^1((0,T), (X_\tau, \|.\|))$, $f_n \to f$ a.e. on $(0, T)$ in $(X, \|.\|)$ and there is an integrable function $\varphi : (0,T) \mapsto IR$ such that $\| f_n(t) \leq \varphi(t)$ pointwise a.e. on $(0, T)$ and $\forall_n \in IN$. Then,

$f \in L^1((0,T), (X_\tau, \|.\|))$, and $\int_0^T \| f_n(t) - f(t) \| dt \to 0$.

Proof.

Since $f_n \to f$ a.e. on (0, T) in (X, ||.||), then $\| f_m - f_n \|$ is Lebesgue measurable and $\| f_m - f_n \| \le 2\varphi$ a.e. on (0, T). Therefore, the Lebesgue theorem we deduce that $\int_0^T \| f_m(t) - f_n(t) \| dt \to 0$, thus, $(f_n)_n$ a Cauchy sequence in $L^1((0,T),(X_\tau, \|.\|))$ which is complete. Consequently, there exists $\tilde{f} \in L^1((0,T),(X_\tau, \|.\|))$ such that $\int_0^T \| f_n(t) - \tilde{f}(t) \| dt \to 0$ and $f_n \to \tilde{f}$ in $(X, \|.\|)$ so $f \equiv \tilde{f}$ a.e. on (0, T).

Remark 4.6

Let $1 \le p \le \infty$. Then spaces $L^p((0,T),(X,\tau))$ equipped with the topology generated by the separating family semi-norms $(A_q^p)_{q\in Q}$ are also sequentially completes and the Lebesgue Theorem could be generalized there under the boundedness notion associated to locally convex topological vector space.

5. Density of Some Suitable Functions Collection in Bochner Spaces

Proposition 5.1

The collection of functions of the form

$$f_n(t) = \sum_{i=1}^n c_i \varphi_i(t),$$

Where $\varphi_i \in C_c^\infty(0,T)$ and $c_i \in X$, are in $L^1((0,T),(X,\tau)) \cap L^1((0,T),(X_\tau, \|.\|))$.

Proof.

Since $\varphi_i \in C_c^\infty(0,T)$ and for any semi-norm $q \in Q$ we have $\forall_t, t' \in (0,T)$

$$q(f_n(t) - f_n(t')) \le \sum_{i=1}^n q(c_i) | \varphi_i(t) - \varphi_i(t') |.$$

Then the collection $(f_n)_n$ are continuous with respect to the topology τ, thus in particular, they are measurable with respect to τ and also Bochner integrable by

$$\int_0^T q(f_n(t))dt \le i = \sum_{i=1}^n q(c_i) \int_0^T |\varphi_i(t)| dt.$$

Remark that the two above inequalities could be hold also by the norm, which claim the result.

Remark 5.1

This collection of functions could not be dense in $L^1((0,T),(X_\tau, \|.\|))$, if not, every function $f \in L^1((0,T),(X_\tau, \|.\|))$ will be a limit of a sequence $(f_n(t))$ with respect to the norm, in this case f will be strongly measurable, hence

$$L^1((0,T),(X_\tau, \|.\|)) \subset \cap L^1((0,T),(X, \|.\|)).$$

By the second example in our introduction we see that it is absurd, but if we equipped the space $L^1((0,T),(X_\tau, \|.\|))$ by the topology generated by the family of semi-norms $(A_q^1)_{q\in Q}$ we can show the density of the above collection in it but under this topology generated by those semi-norms.

Let recall that whenever $f \in L^1((0,T),(X_\tau, \|.\|))$, then, there exists $(K_n)_n$ a sequence of compacts such that $\mu(I \setminus K_n) \to 0$ and $f \mid K_n$ is continuous in (X, τ). Put $f_n = f\chi K_n$ where χK_n the characteristic function on K_n. So, $f_n \to f$ in $L^1((0,T),(X_\tau, \|.\|))$ with respect to the norm $\|.\|_1$, hence with respect to the topology generated by the family of semi-norms $(A_q^1)_{q\in Q}$ by the lemma (4.1).

Definition 5.1

A vector-valued function f defined on a measurable set E of I = (0, T) with values in X is said to be finitely vector-valued if there is a finite disjoint class $\{E_1,\ldots, E_n\}$ of measurable sets and a finite set $\{c_1,\ldots, c_n\}$ of vector points such that

$$f(t) = \sum_{i=1}^n c_i x E_i(t).$$

Remark 5.2

According to our definition of measurability with respect to the topology τ, then every finitely vector-valued function $f: I=(0,T)\mapsto(X,\tau)$ is measurable with respect to τ.

Denote

$\theta(I,X)$: The class of finitely vector-valued functions acting from I to X.

Lemma 5.1

Let $f:(0,T)\mapsto(X,\tau)$ an uniformly continuous function on a compact $K\subset(0,T)$ and $f\equiv 0$ on K^c. Then there exists a sequence $(f_n)_n$ of elements of $\theta(I,X)$ converging to f with respect to $(A_q^1)_{q\in Q}$.

Proof.

Since f is uniformly continuous on the compact $K\subset(0,T)$ and choosing a n- subdivisions of K $E_i^n=[t_{i-1}^n,t_i^n]$ with $t_i^n=t_{i-1}^n+\dfrac{T}{n},\xi_i^n\in E_i^n, i=1,\dots,n$ and put

$$f_n(t)=\sum_{i=1}^{n} f(\xi_i^n)\chi_{E_i}^n(t).$$

Then $(f_n)\in\theta(I,X)$. Let us prove now that f_n converges to f with respect to the topology generated by $(A_q^1)_{q\in Q}$.

Let $\varepsilon>0$, $q\in Q$ and since $f:K\mapsto(X,\tau)$ uniformly continuous, we deduce that there is a η such that

$$\forall_t, t'\in K, |t-t'|<\eta\Rightarrow q(f(t)-f(t'))<\varepsilon.$$

Further, let $n_0\in IN$ such that $\forall_n\geq n_0, \dfrac{T}{n}<\eta$, and $t\in K$ there exists i such that $1\leq i\leq n$ and $t\in E_i^n$. Therefore, $f_n(t)=f(\xi_i^n)$ and $|t-\xi_i^n|<\eta$. It follows then that

$$q(f_n(t)-f(t))=q(f(t)-f(\xi_i^n))<\varepsilon.$$

Consequently, f_n converges to f on K with respect to the topology τ : Moreover, $\forall_q\in Q,\forall_t\in I,\forall_n\in IN,\exists M_q$ such that $q(f_n(t))\leq= M_q$.

Finally, by the dominated convergence theorem we conclude that f_n converges to f with respect to the topology generated by $(A_q^1)_{q\in Q}$.

Theorem 5.1

The collection of function of the form:

$$f(t)=\sum_{i=1}^{n} c_i\varphi_i(t),$$

Where $\varphi_i\in^{\infty}_{Cc}(0,T)$ and $c_i\in X$, are dense in $L^1((0,T),(X_\tau,\|.\|))$ under the topology generated by $(A_q^1)_{q\in Q}$.

Proof.

Consider simple functions of the form:

$$f(t)=\sum_{i=1}^{n} c_i\chi E_i(t)$$

And since $\varphi_i\in C_c^\infty(0,T)$ is dense in $L^1((0,T),IR$, then the characteristic function $\chi E_i(t)$ could be mollifying by a functions $\varphi_i\in C_c^\infty(0,T)$. Hence, the collection

$$f(t)=\sum_{i=1}^{n} c_i\varphi_i(t),$$

are dense in $\theta(I,X)$ under the topology generated by

$(A_q^1)_{q\in Q}$. Thus, by the lemma (5.1) and remark (5.1) we have that $\theta(I,X)$ is dense in $L^1((0,T),(X_\tau,\|.\|))$ with respect to $(A_q^1)_{q\in Q}$, which implies the wanted result.

Corollary 5.1

If the topological vector space X is separable with respect to τ, then the space $L^1((0,T),(X_\tau,\|.\|))$ is separable too under the topology generated by $(A_q^1)_{q\in Q}$.

Proof.

Since (X,τ) and the space $\varphi_i\in C_c^\infty(0,T)$ are separable, then, by the theorem (5.1) the result is immediate.

Remark 5.3

The collection $f(t)=\sum_{i=1}^{n} c_i\varphi_i(t)$ with $\varphi_i \in C_c^\infty(0,T)$ and $c_i \in X$, are dense also in the $L^1((0,T),(X,\tau))$ under the topology generated by $(A_q^1)_{q\in Q}$. Thus it is separable also if (X,τ) is separable.

Remark 5.3

The collection $f(t)=\sum_{i=1}^{n} c_i\varphi_i(t)$ with $\varphi_i \in C_c^\infty(0,T)$ are most important and useful as a Galerkin approximations in the variational formulation of Navier Stokes equations or as a Shauder basis for suitable locally convex topological vector spaces (see P. K. Kamthan [6]).

6. Differentiability and Bochner-Sobolev Spaces

6.1 Strong Differentiability Notion

Definition 6.1

Let (X,τ) is any locally convex topological vector space. A function $f:(0,T)\mapsto(X,\tau)$ is strongly di erentiable at every point of $(0,T)$, with strong point wise derivative $f'(t)$, if it is continuous and

$$f'(t)=\lim_{h\to 0}[\frac{f(t+h)-f(t)}{h}]$$

where the limit exists with respect to each semi-norm on X.

In the case that X is a normed space we take the above limit with respect to the norm, and we say that f is continuously differentiable in $(0,T)$ if its point wise derivative exists for every $t\in(0,T)$ and $f':(0,T)\mapsto(X,\tau))$ is a continuous.

Remark 6.1

The assumption of continuous differentiability is often too strong to be useful or suitable to resolve some differential equations as the requirement that the strong point wise derivative exists a.e. on $(0,T)$ does not lead to an effective theory. Instead, we use the notion of a distributional or weak derivative.

Remark 6.1

To deal with weak differentiability for vector-valued function, the assumption that the space should be separable will be most fundamental, but there are two cases which are either X is separable with respect to the topology τ or to the norm. In this axis, we can see by the lemma (4.1) that if the space X is separable with respect to the norm it is separable also with respect to the topology τ. This assumption of separability of the space X will be our tool to deduce the Lebesgue differentiation theorem for vector-valued function and since the norm could be seen as a semi-norm, then we will be concerned here with the weak differentiability with respect to the topology τ in the both spaces $L^1((0,T),(X,\tau))$ and $L^1((0,T),(X_\tau,\|.\|))$.

6.2 Weak Differentiability Notion

Let (X,τ) be a locally convex topological vector space and separable with respect to the topology τ.

Definition 6.2

A function $f\in L^1((0,T),(X,\tau))$ is weakly differentiable with weak derivative $f'=\frac{df}{dt}=g\in L^1((0,T),(X,\tau))$ if for every $\phi\in C_c^\infty(0,T)$ we have

$$\int_0^T \phi' fdt=-\int \phi g dt.$$

The above integrals are understood to be Bochner integrals.

Recall that if $f:(0,T)\mapsto(X,\tau)$ is a scalar-valued integrable function, then the Lebesgue differentiation theorem implies that the limit

$$\lim_{h\to 0}\frac{1}{h}\int_t^{t+h} f(s)ds$$

exists and is equal to $f(t)$ for t point wise a.e. on $(0,T)$.

Theorem 6.1

Let $f\in L^1((0,T),(X,\tau))$, then

$$f(t)=\lim_{h\to 0}\frac{1}{h}\int_t^{t+h} f(s)ds$$

for pointwise a.e. on $(0,T)$, the limit is taken with respect to the topology τ.

Remark 6.3

This theorem could be hold in the space $L^1((0,T),(X_\tau,\|.\|))$, but the limit of Lebesgue differentiation is taken always with respect to the topology τ unless if the X is separable with respect to the norm.

Proof.

Let $\{c_n \in X, n\in IN\}$ be a dense subset of (X,τ), then by Lebesgue differentiation theorem for scalar-valued functions , for any semi-norm $q\in Q$ we have for any $n\in IN$ and t point wise a.e. on $(0,T)$:

$$q(f(t)-c_n)=\lim_{h\to 0}\frac{1}{h}\int_t^{t+h} q(f(s)-c_n)ds$$

since for all $f\in L^1((0,T),(X,\tau))$, $qof \in L^1(0,T)$. Thus, for all such $t\in(0,T)$

$$\begin{aligned}&\lim_{h\to 0}\frac{1}{h}\int_t^{t+h} q(f(s)-f(t))ds\\ &\le \lim_{h\to 0}\frac{1}{h}[\int_t^{t+h} q(f(s)-c_n)ds+\int_t^{t+h} q(f(t)-c_n)ds]\\ &\le 2q(f(t)-c_n),\end{aligned}$$

since this hold for every c_n it follows that

$$\lim_{h\to 0}\frac{1}{h}\int_t^{t+h} q(f(s)-f(t))ds=0.$$

Therefore,

$$\lim_{h\to 0} q(\frac{1}{h}\int_t^{t+h} f(s)ds-f(t))\le \lim_{h\to 0}\frac{1}{h}\int_t^{t+h} q(f(s)-f(t))ds=0$$

which proves the result.

Remark 6.4

By the Pettis theorem every strongly measurable function should be almost separably valued so the assumption that the space X should be separable is necessary to prove the Lebesgue differentiation theorem for vector-valued functions. Hence we could ask if measurable function with respect topology are almost separably valued too.

Lemma 6.1

Suppose that $f:(0,T)\mapsto(X,\tau)$ is a locally Bochner integrable function such that for every $\phi_i\in C_c^\infty(0,T)$

$$\int_0^T \phi f dt=0,$$

Then $f\equiv 0$ point wise a.e. on $(0,T)$.

Proof.

Take the characteristic function $\chi(t,t+h)$ of the interval $(t,t+h)\subset(0,T)$, then $\chi(t,t+h)\in L^1((0,T)),IR$. So, there exists a sequence of test functions $\phi_n\in C_c^\infty(0,T)$, without loss of generality $\phi_n\le 1$ and are contained inside a fixed compact subset of $(0,T)$ such that $\phi_n\to\chi(t,t+h)$ point wise a.e. on $(0,T)$. Now, pick any $s\in(0,T)$ and $q\in Q$, then

$$\begin{aligned}&q(\phi_n(s)f(s)-\chi(t,t+h)(s)f(s))\\ &\le|\phi_n(s)-\chi(t,t+h)(s)|\,q(f(s)).\end{aligned}$$

Thus, $\phi_n f\to\chi(t,t+h)f$ point wise a.e. in (X,τ), and since $f\in L^1_{loc}((0,T),(X,\tau))$, then by the dominated convergence theorem

$$\int_t^{t+h} f(s)ds=\lim_{h\to 0}\int_t^{t+h}\phi_n f(s)ds,$$

where the limit is taken with respect to τ. Finally, if $\int_0^T \phi f dt=0$ for every $\phi\in C_c^\infty(0,T)$, then $\int_t^{t+h} f(s)ds=0$ for every $(t,t+h)\subset(0,T)$, so by the Lebesgue differentiation theorem it follows that $f\equiv 0$ point wise a.e. on $(0,T)$.

Proposition 6.1

Suppose that $f:(0,T)\mapsto(X,\tau)$ is weakly differentiable and $f'=0$, then f is equivalent to a constant function.

Proof.

Choose $\phi \in C_c^\infty(0,T)$ such that $\int_0^T \phi = 1$, then for every $\varphi \in C_c^\infty(0,T)$ the function $\eta(t) = \int_0^t [\varphi(s) - (\int_0^T \varphi)\phi(s)]ds$ is a function in $C_c^\infty(0,T)$. Hence, by assumption and by definition of the weak derivative we obtain $\forall_\varphi \in C_c^\infty(0,T)$

$$\int_0^T f(\varphi - (\int_0^T \varphi)\phi)ds = -\int_0^T f'\eta = 0$$

If we put $c = \int_0^T f\phi \in X$, then the previous equality becomes for every $\varphi \in C_c^\infty(0,T)$

$$\int_0^T (f - c)\varphi = 0$$

so, by the lemma (6.1) we conclude that $f = c$ point wise a.e. on $(0,T)$.

Remark 6.5

Let (X,τ) a locally convex topological vector space and consider the initial value problem of first differential equation:

$$x'(t) = f(t), t \in (0,T), x(0) = c_0$$

where the vector-valued function $f \in L^1((0,T),(X,\tau))$. Our goal is to know if this problem has a solution and in which space could be exists?

Theorem 6.2

Suppose that $f \in L^1((0,T),(X,\tau))$, then f is weakly differentiable with integrable derivative $f' = g \in L^1((0,T),(X,\tau))$ if and only if

$$f(t) = c_0 + \int_0^t g(s)ds$$

point wise a.e. on $(0,T)$ with c_0 any constants in X. In that case, f is strongly differentiable point wise a.e. and its point wise derivative coincides with its weak derivative.

Proof.

Suppose that f is given by $f(t) = c_0 + \int_0^t g(s)ds$ point wise a.e. on $(0,T)$ for $f \in L^1((0,T),(X,\tau))$, then $\dfrac{f(t+h) - f(t)}{h} = \dfrac{1}{h}\int_t^{t+h} g(s)ds$

so, by the Lebesgue differentiation theorem we conclude that the strong derivative of f with respect to τ exists a.e. and is equal to g, we also have that for $q \in Q$ any semi-norm on X

$$q(\frac{f(t+h) - f(t)}{h}) \le \frac{1}{h}\int_t^{t+h} q(g(s))ds.$$

Extending f by zero to a function $f: IR \mapsto X$, using Fubini's theorem and Lebesgue differentiation theorem we get

$$\int_{IR} q(\frac{f(t+h) - f(t)}{h})dt \le \frac{1}{h}\int_{IR}(\int_t^{t+h} q(g(s))ds)dt$$

$$\le \frac{1}{h}\int_0^h (\int_{IR} q(g(s+t))dt)ds \le \int_{IR} q(g(t))dt.$$

Now, we need to show that g is also the weak derivative of f. Let $\phi \in C_c^\infty(0,T)$, then if we use the dominated convergence theorem and the previous result on the point wise a.e. convergence of f' we get

$$\int_0^T \phi'(t) f(t)dt = \lim_{h\to 0}\int_0^T [\frac{\phi(t+h) - \phi(t)}{h}] f(t)dt$$

$$= -\lim_{h\to 0}\int_0^T \phi(t)[\frac{f(t) - f(t-h)}{h}]dt = -\int_0^T \phi(t)g(t)dt.$$

Conversely, if $f \in L^1((0,T),(X,\tau))$ and and its weak derivative $f' = g \in L^1((0,T),(X,\tau))$.

Take

$$\tilde{f}(t) = \int_0^t g(s)ds$$

so, similarly with previous argument we show that $\tilde{f} \in L^1((0,T),(X,\tau))$ and its weak derivative is $\tilde{f}' = g$, then $(f - \tilde{f})'$ is zero, by the proposition (6.1) we obtain $f - \tilde{f} = c$ point wise a.e. which claim the result.

6.3 Bochner-Sobolev Spaces

Let (X,τ) a locally convex topological vector space and sequentially complete and let $1\le p\le\infty$. Then we define Bochner-Sobolev space as follow:

$W^{1,p}((0,T),(X,\tau))$:

$\{f\in L^p((0,T),(X,\tau))\mid g\in L^p((0,T),(X,\tau))$

such that for every $\phi\in C_c^\infty(0,T)$ one has $\int_0^T \phi' f dt=-\int_0^T \phi g dt\}$.

The function g is uniquely determined, if it exists, to see this, it is sufficient to apply the proposition (6.1).

Let $W^{1,p}((0,T),(X,\tau))$ and consider the following family of semi-norms such that:

$W_q^{1,p}(f)=A_q^p(f)+A_q^p(f')$

Where $q\in Q$.

Proposition 6.2

The Bochner-Sobolev space $W^{1,p}((0,T),(X,\tau))$ equipped by the family of semi-norms $(W_q^{1,p})_{(q\in Q)}$ is a sequentially complete.

Proof.

It is an immediate result of the fact that $L^p((0,T),(X,\tau))$ is sequentially complete and the use of dominated convergence theorem.

Remark 6.6

We can also define other Bochner-Sobolev spaces when $(X,\tau,\|.\|)$ is a bi-topological space as follow:

$W^{1,p}((0,T),(X_\tau,\|.\|))$:

$\{f\in L^p((0,T),(X_\tau,\|.\|))\mid g\in L^p((0,T),(X_\tau,\|.\|))$

such that for every $\phi\in C_c^\infty(0,T)$ one has $\int_0^T \phi' f dt=-\int_0^T \phi g dt\}$.

Let $f\in W^{1,p}((0,T),(X_\tau,\|.\|))$ and consider the norm:

$$\|f\|_{W^{1,p}}=\|f\|_{L_p}+\|f'\|_{L_p}$$

under the above norm, the space $W^{1,p}((0,T),(X_\tau,\|.\|))$ is a Banach space.

Corollary 6.1

Let $t_0\in(0,T)$, $g\in L^1((0,T),(X,\tau))$ and set for every $t\in(0,T)$

$$f(t)=\int_{t_0}^t g(s)ds.$$

Then $W^{1,1}((0,T),(X,\tau)$ and $f'=g$.

Proof.

It is an immediate consequence of the theorem (6.2).

Theorem 6.3

Let $f\in W^{1,1}((0,T),(X,\tau)$. Then there exists a continuous function $\tilde f:(0,T)\mapsto(X,\tau))$, which coincides with f almost everywhere and such that for every s, $t\in(0,T)$

$$\tilde f(t)-\tilde f(s)=\int_s^t f'(r)dr.$$

Proof.

Fix $t_0\in(0,T)$ and set $h(t):=\int_{t_0}^t f'(s)ds$ for every $t\in(0,T)$, then, since the Lebesgue integral is absolutely continuous, clearly, the function h is continuous in (X,τ). By the corollary (6.1) we get $h\in W^{1,1}((0,T),(X,\tau)$ and $h'=f'$. So by the proposition (6.1) we have $f-h=c$ a.e. for some constant $c\in X$. Set $\tilde f=h+c$. Then f coincides almost everywhere with the continuous function $\tilde f$, and that

$$\tilde f(t)-(\tilde f(s)=h(t)-h(s)=\int_s^t f'(r)dr.$$

Remark 6.7

Due to this theorem, we identify every function $f\in W^{1,1}((0,T),(X,\tau)$ with its continuous representative $\tilde f$, and we could say that every function $f\in W^{1,1}((0,T),(X,\tau)$ is continuous with respect to τ.

Remark 6.8

In the case that $f \in W^{1,p}((0,T),(X_\tau,\|.\|))$, the continuous representative of f is uniformly continuous from $(0,T)$ to $(X,\|.\|)$ but could be extended just to a continuous function from $\tilde{I}=(0,T)$ to (X,τ) since X is supposed sequentially complete under the topology τ and not to the norm.

Indeed, we have for any

$t,s \in (0,T),\ f(t)-f(s)=\int_s^t f'(r)dr$, then

$\| f(t)-f(s)\| = \int_s^t \| f'(r)\| dr.$

Since $\| f\| \in L^1(0,T)$, and the Lebesgue integral is absolutely continuous, thus f is uniformly continuous on $(0,T)$ with respect to the norm. But if we consider the sequence $f_n = f(\frac{1}{n}) \in X, n \in IN$, then we deduce that $(f_n)_n$ is a Cauchy sequence in $(X,\|.\|)$, so it is also a Cauchy sequence in (X,τ) which assumed sequentially complete, hence converges to a certain limit denoted $f(t)$. Therefore, the function f could be extended to a continuous function $f:[0,T]\mapsto (X,\tau)$.

Theorem 6.4

(Sobolev Embedding Theorem)

Let $1 \le p \le \infty$ and $W^{1,p}((0,T),(X,\tau))$, then $f \in C([0,T]\mapsto (X,\tau))$ and there exists a constant $C=C(p,T)$ such that

$A_q^\infty(f) \le C W_q^{1,p}(f)$

for every semi-norm $q \in Q$.

Proof.

Let $q \in Q$, then from theorem (6.3), we get for every $t,s \in (0,T)$

$$q(f(t)-f(s)) \le \int_s^t q(f'(r))dr$$

and if $h:(0,T)\mapsto IR$ is defined by $h=q(f)$, then

$$|h(t)-h(s)| \le q(f(t)-f(s)) \le \int_s^t q(f'(r))dr,$$

it follows that h is absolutely continuous and by the Lebesgue differentiation theorem for real-valued functions we get $|h'| \le q(f')$ point wise a.e. on $(0,T)$. Therefore, by the Sobolev embedding theorem for real-valued functions we obtain

$A_q^\infty(f) = \| h\|_\infty \le C \| h\|_{W^{1,p}(0,t)} \le C W_q^{1,p}(f).$

Remark 6.9

For spaces $W^{1,p}((0,T),(X_\tau,\|.\|))$, by the same as above we can find a constant $C=C(p,T)$ such that for every $f \in W^{1,1}((0,T),(X,\tau))$ we have

$\| f\|_\infty \le C \| h\|_{W^{1,p}}.$

Remark 6.10

We can prove the Sobolev embedding theorem also by considering the identity mapping:

$W^{1,p}((0,T),(X,\tau)) \mapsto C([0,T],(X,\tau))$
$f \mapsto f$

in which the space $C([0,T],(X,\tau))$ will be equipped by the family of semi-norms

$\left\{ \sup_{t\in[0,T]} q(f(t)); q \in Q \right\}$

for any $f \in C([0,T],(X,\tau))$. Since

$f(t)-f(s)=\int_s^t f'(r))dr$

then, if a sequence $(f_n)_n$ is convergent in $W^{1,p}((0,T),(X,\tau))$, it converges also in $C([0,T],(X,\tau))$. So, let $f_n \to f$ in $W^{1,p}((0,T),(X,\tau))$ and $f_n \to g$ in $C([0,T],(X,\tau))$, and since both spaces are continuously embedded into $L^p((0,T),(X,\tau))$ then $f_n \to f$ and $f_n \to g$ in $L^p((0,T),(X,\tau))$ Hence, by uniqueness of the limit we obtain that $f=g$ and the identity mapping is closed.

Proposition 6.3 “CONVOLUTION”

Let $\phi \in L_1(IR)$ and $f \in L^p((0,T),(X,\tau))$ where $1 \le p \le \infty$. Then for a.e. $t \in IR$, the function $s \mapsto \phi(t-s)f(s)$ is Bochner integrable on IR. Set

$$(\phi * f)(t) = \int_{IR} \phi(t-s)f(s)ds$$

thus, $\phi * f \in L^p((0,T),(X,\tau))$ and for any semi-norm $q \in Q$

$$A_q^p(\phi * f) \le \|\phi\|_{L^1} \; A_q^p(f).$$

Proof.

We adopt the same argument as theorem (IV.15) in H. Brezis [1].

Proposition 6.4

Let $\phi \in C_c^k(IR)$ and $f \in L^1_{loc}(IR,(X,\tau))$. Then $\phi * f \in C^k(IR,(X,\tau))$ and $D^n(\phi * f) = D^n(\phi) * f$.

In particular, if $\phi \in C_c^\infty(IR)$ and $f \in L^1_{loc}(IR,(X,\tau))$ Then

$$\phi * f \in C^\infty(IR,(X,\tau)).$$

Proof.

See proposition (IV.20) in H. Brezis [7].

Theorem 6.5 "MOLLIFICATION"

Let $f \in W^{1,p}((0,T),(X,\tau)$ extended by zero outside of $I=(0,T)$ and $\omega(.)$ the standard mollify denote

$$f_h(t) = \frac{1}{h} \, f \int_{IR} \omega(\frac{t-s}{h}) f(s)ds.$$

Then:

i) $f_h(t) \in C^\infty([0,T],(X,\tau)).$

ii) for $1 \le p \le \infty$ we have $f_h \to f$ in $L^p((0,T),(X,\tau))$.

iii) for $1 \le p \le \infty$ we have $f_h' \to f'$ in $L^p((0,T),(X,\tau))$.

iv) for $1 \le p \le \infty$ and $q \in Q$ a semi-norm we get

$$A_q^p(f_h) \le A_q^p(f) \quad and \quad A_q^p(f'_h) \le A_q^p(f').$$

Proof.

Since the standard mollify function is taken always $\omega \in C_c^\infty(IR)$ with support contained in $(-h,h)$ such that

$$f_{\omega h} = \frac{1}{h} \, f \int \omega(\frac{s}{h})ds = 1.$$

Then using the propositions (6.3) and (6.4) also the dominated convergence theorem we achieve the proof.

Theorem 6.6

(Product rule, Integration by parts)

Let $f \in W^{1,p}((0,T),(X,\tau)$ and $g \in W^{1,p}((0,T),IR)$.

Then for every $1 \le p \le \infty$, we have

i) (Product rule): The product f_g belongs to $W^{1,p}((0,T),(X,\tau)$ and $(fg)' = f'g + fg'$.

ii) (Integration by parts)

$$f_0^T f'_g = f(T)g(T) - f(0)g(0) - \int_0^T fg'.$$

Proof.

By the two theorems (6.4), (6.5) and we adopt the same argument as in the proof of the corollary (VIII.9) in H. Brezis [7] we claim our result.

Definition 6.3

Let $1 \le p \le \infty$ and define $W_0^{1,p}((0,T),(X,\tau)$ as a closing subset of $C_c^1((0,T),(X,\tau)$ in $W_0^{1,p}((0,T),(X,\tau)$ with respect to the topology generated by the family of semi-norms $(W_q^{1,p})_{(q \in Q)}$.

Theorem 6.7

Let $f \in W^{1,p}((0,T),(X,\tau))$. Then f belongs to $W_0^{1,p}((0,T),(X,\tau)$ such that $f_n \to f$ in $W^{1,p}((0,T),(X,\tau))$. So, by the theorem (6.4) $f_n \to f$ uniformly on $(0,T)$ with respect to the topology τ .Hence, $f(0) = f(T) = 0$

Conversely, suppose that $f \in W^{1,p}((0,T),(X,\tau))$ and $f(0) = f(T) = 0$. Let

$$\varphi_n(t) : [\frac{1}{n}, T - \frac{1}{n}] \mapsto [0, T]$$

$$t \mapsto \frac{T(t - \frac{1}{n})}{T - \frac{2}{n}}$$

And set

$$h_n(t) = \begin{cases} f(\varphi_n(t)) & if \quad t \in [\frac{1}{n}, T - \frac{1}{n}] \\ 0 & outside. \end{cases}$$

Since $h_n \in W^{1,p}((0,T),(X,\tau)) \cap C_c((0,T),(X,\tau))$. Then by convolution product and mollification, see theorem (6.5), we obtain that $h_n \in W_0^{1,p}((0,T),(X,\tau))$.

In the other hand $h_n \to f$ in $W_0^{1,p}((0,T),(X,\tau))$.

Thus, we achieve the proof.

Theorem 6.8 (Poincare's Inequality)

Let $1 \le p \le \infty$. Then, there exists a constant $C = C(T)$ such that for every $f \in W^{1,p}((0,T),(X,\tau))$ and $q \in Q$ we have

$$f_{\text{o}}^{T} q(f)^p \le C \int_0^T q(f')^p.$$

Proof.

By theorem (6.3), for $f \in W^{1,p}((0,T),(X,\tau))$ we have $q(f(t)) = q(f(t) - f(0)) \le \int_0^t q(f'(t))dt$, thus, using Holder inequality we claim the result.

References

[1] B.J. Pettis, On integration in vector spaces, Transactions of the American Mathematical Society. Amer. Math. Soc., Providence, RI, ISSN 0002-9947, vol. 44 (1938), pp. 277-304.

[2] V.M. Kadets, On the Riemann integrability of weak continuous functions, Quaest. Math. Quaestiones Mathematicae, Journal of the South African Mathematical Society. NISC, Grahamstown, ISSN 1607-3606, vol.17 (1994), no. 1, pp. 33-35. MR 127 6006 (95e: 46056).

[3] A. Alexiewicz, W. Orlicz, Remarks on Riemann-Integration of vector -valued functions, Studia Math. Studia Mathematica. Inst. Math., Polish Acad. Sci., Warsaw, ISSN 0039-3223, vol. 12 (1951), pp. 125-132. MR 0043366 (13, 250c).

[4] Chonghu WANG, Zhenhua YANG, Some topological properties of Banach spaces and Riemann integration, Rocky Mountain J. Math, The Rocky Mountain Journal of Mathematics. Rocky Mountain Math. Consortium, Tempe, AZ. ISSN0035-7596, vol. 30 (2000), no. 1, pp. 393-400. MR 1763820(2001d: 46062).

[5] D.H. Hyers, Pseudonormed Linear Spaces and Abelian Groups, Duke Math. J. Duke Mathematical Journal. Duke Univ. Press, Durham, NC, ISSN 0012-7094, vol. 5 (1939), pp. 628-634.

[6] P.K. Kamthan, Normalized Bases in Topological Vector, Port. Math. Portugaliae Mathematica, A Journal of the Portuguese Mathematical Society. Eur. Math. Soc., Z¨urich, ISSN 0032-5155, vol. 38, Fasc. 3-4 (1979).

[7] H. Brezis, Analyse Fonctionelle Théorie et Application, Masson, Paris, ISSN 0754-4405 (1987).