Journal of Mathematical Sciences 2 (2015) 171-184
ISSN 2372-5214
Betty Jones & Sisters Publishing
http://www.bettyjonespub.com

P-adic Zeta Functions and Quasi-modular Forms

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Abstract: For an algebraic group G over a number field K these L functions (or simply zeta-functions in Shimura's terminology) are defined as certain Euler products denoted by $L(s,\pi,r,\chi)$. We present a method to construct a *p*-adic analogue of $L(s,\pi,r,\chi)$ which is a *p*-adic analytic function $L_p(s,\pi,r,\chi)$ of *p*-adic arguments $s\in Z_p, \chi$ mod p^r interpolating algebraic numbers defined through the normalized critical values $L^*(s,\pi,r,\chi)$ of the corresponding complex analytic *L*-function. Fourier coefficients of general quasi-modular forms are used.

Keywords: Siegel modular forms, *L*-functions, *p*-adic modular forms, quasi-modular forms.

1. Introduction

Automorphic *L*-function (or simply zeta-functions in Shimura's terminology, see [1]) can be attached to an algebraic group G of symplectic or unitary type (Sp, UT or UB) over a number field K, as certain Euler products denoted by $L(s,\pi,r,\chi)$.

A *p*-adic analogue of $L(s,\pi,r,\chi)$ is a *p*-adic analytic function $L_p(s,\pi,r,\chi)$ of *p*-adic arguments $s\in Z_p, \chi$ mod p^r which interpolates algebraic numbers defined through the normalized critical values $L^*(s,\pi,r,\chi)$ of the corresponding complex analytic *L*-function. We present a method using arithmetic nearly-holomorphic forms and general quasimodular forms, related to algebraic automorphic forms. It gives new technique of constructing *p*-adic zeta-functions via general quasi-modular forms and their Fourier coefficients.

2 Automorphic *L*-functions and Their *P*-adic Analogues

Main objects in this paper are automorphic *L*-functions and their *P*-adic analogues.

For an algebraic group G over a number field K these L-functions (or simply zeta-functions in Shimura's terminology) are defined as certain Euler products.

Example 1.1 $(G=GL(2), K=Q, L_f(s)=\sum_{n\geq 1} a_n n^{-s}, s\in C)$. Here $\sum_{n\geq 1} a_n q^n$ is a modular form on the upper-half plane

$$H=\{z\in C, \mathrm{Im}(z)>0\}=SL(2)/SO(2), q=e^{2\pi i z}.$$

An Euler product has the form

$$L_f(s)=\prod_{p\ primes}(1-a_p p^{-s}+\varphi_f(p)p^{k-1-2s})^{-1}$$

where *k* is the weight and ψ_f the Dirichlet character of f. It is defined *iff* the representation π_f attached to f is irreducible.

Recall that π_f is generated by the lift $\tilde{f}$ of f to the group $G(A)$, where A is the ring of adeles

$$A=\{x=(x_\infty,x_p)_p \mid x_\infty\in IR, x_p\in Q_p,$$

such that $x_p\in Z_p$ for all but a finite number of p }.

2.1 A P-adic Analogue of L_f (s) (Manin-Mazur)

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It is a p-adic analytic function $L_{f,p}(s,\chi)$ of p-adic arguments $s \in Z_p, \chi$ mod p^r which interpolates algebraic numbers

$$L_f^*(s,\chi)/\omega^{\pm} \in \overline{Q} \to C_p = \hat{\overline{Q}}_p \text{ (the Tate field)}$$

for $1 \le s \le k-1$, $\omega^{\pm}$ are periods of f where the complex analytic L function of f is defined for all $s \in C$ so that in the absolutely convergent case $\mathrm{Re}(s) > (k+1)/2,\ L_f^*(s,\chi) = (2\pi)^{-s}\Gamma(s)\sum_{n\ge 1}\chi(n)a_n n^{-s}$, which extends to holomorphic function with a functional equation. According to Manin and Shimura, this number is algebraic if the period $\omega^{\pm}$ is chosen according to the parity $\chi(-1)(-1)^{-s} = \pm 1$.

2.2 Constructions of P-adic Analogues

In general case of an irreducible automorphic representation of the adelic group $G(A_k)$ there is an L-function

$$L(s,\pi,r,\chi) = \prod_{p_v\ primes \in K}\prod_{j=1}^{m}(1-\beta_{j,p_v}N_{p_v^{-s}})^{-1}$$

where

$$\prod_{j=1}^{m}(1-\beta_{j,p}X) = \det(1_m - r(diag(a_{i,p})_i X)),$$

$\alpha_{i,p}$ are the Satake parameters of $\pi = \otimes_v \pi_v v \in \sum_K (place\ in\ K), p = p_v$.

Here $h_v = diag(\alpha_{i,p})_i$ live in the Langlands group ${}^L G(C), r{:}{}^L G(C) \to GL_m(C)$ denotes its representation, $\chi : A_K^*/K^* \to C^*$ is a character of finite order. Constructions admit extention to rather general automorphic representations on Shimura varieties via the following tools:

- Modular symbols and their higher analogues (linear forms on cohomology spaces related to automorphic forms);
- Petersson products with a fixed automorphic form;
- Linear forms coming from the Fourier coefficients (or Whittaker functions), or throught the
- CM-values (special points on Shimura varieties).

2.3 Accessible Cases: Symplectic and Unitary Groups

- $G = GL_1$ over Q (Kubota-Leopoldt-Mazur) for the Dirichlet L-function $L(s,\chi)$;
- $G = GL_1$ over a totally real field f (Deligne-Ribet, using algebraicity result by Klingen;
- $G = GL_1$ over a CM-field K, i.e. a totally imaginary extension of a totally real field f (N. Katz, Manin-Vishik);
- the Siegel modular case $G = GSp_n$ (the Siegel modular case, $F = Q$);
- General symplectic and unitary groups over a CM-field K.

3 Automorphic *L*-functions Attached to Symplectic and Unitary Groups

Let us briefly describe the L-functions attached to symplectic and unitary groups as certain Euler products in Chapter 5 of [1], with critical values computed in Chapter 7, Theorem 28.8 using general nearly holomorphic arithmetical automorphic forms.

$$G = G(\varphi) = \{\alpha \in GL_m(K) \mid \alpha\varphi^t\alpha^{\rho} = v(\alpha)\varphi\},$$
$$v(\alpha) \in F^*,$$

where $\varphi = \eta_n = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$ or $\varphi = \begin{pmatrix} 1_n & 0 \\ 0 & 1_m \end{pmatrix}$,

see Ch. Skinner and E. Urban [2] and Shimura G., [1].

3.1 The Groups and Automorphic Forms Studied in Shimura's Book

Let F be a totally real algebraic number field, K be a totally imaginary quadratic extension of F and ρ be the generator of $Gal(K/F)$. Take $\eta_n = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$ and define

$G = Sp(n,F)$ (Case Sp)

$G = \{\alpha \in GL_{2n}(K) \mid \alpha\eta_n\alpha^* = \eta_n\}$ (Case UT = unitary tube)

$G = \{\alpha \in GL_{2n}(K) \mid \alpha T\alpha^* = T\}$ (Case UB = unitary ball).

According to three cases. Assume $F = Q$ for a while. The group of the real points G_∞ acts on the associated domain

$$H = \begin{cases} \{z \in M(n,n,C) \mid {}^t z = z, \mathrm{Im}(z) > 0\} & \text{(Case Sp)} \\ \{z \in M(n,n,C) \mid i(z^* - z) > 0\} & \text{(Case UT)} \\ \{z \in M(n,n,C) \mid 1_q - z^* z > 0\} & \text{(Case UB)} \end{cases}$$

$(p,q), p+q=n$ being the signature of iT . Here $z^* = {}^t\bar{z} = z$ and $>$ means that a hermitian matrix is positive definite. In Case UB, there is the standard automorphic factor $M(g,z), g \in G_\infty, z \in H$ taking values in $GL_p(C) \times GL_q(C)$.

3.2 Shimura's Arithmeticity in the Theory of Automorphic Forms [1], P-adic Zeta Functions and Nearly-holomorphic Formson Classical Groups

Automorphic *L*-functions via general quasimodular forms: Automorphic *L*-functions and their *p*-adic analogues can be obtained for quite general automorphic representations on Shimura varieties by constructing *p*-adic distributions out of algebraic numbers attached to automorphic forms. These numbers satisfy certain Kummer-type congruences established in different ways via:

- Normalized Petersson products with a fixed automorphic form;
- Linear forms coming from the Fourier coefficients (or Whittaker functions);
- CM-values (special points on Shimura varieties), see The Iwasawa Main Conjecture for $GL(2)$ by $C.$Skinner and E. Urban, [2], Shimura G., Arithmeticity in the theory of automorphic forms [1].

The combinatorial structure of the Fourier coefficients of the holomorphic forms used in these constructions is quite complicated.

In order to prove the congruences needed for the *p*-adic constructions, we use a simpli_cation due to nearly-holomorphic and general quasimodular forms, related to algebraic automorphic forms. In this paper, a new method of constructing *p*-adic zeta-functios is presented using general quasi-modular forms and their Fourier coefficients.

In order to describe both algebraicity and congruences of the critical values of the zeta functions of automorphic forms on unitary and symplectic groups, we follow the review by H. Yoshida [3] of Shimura's book "Arithmeticity in the theory of automorphic forms" [1]. Shimura's mathematics developed by stages:

(A) Complex multiplication of abelianvarieties = >;

(B) The theory of canonical models = Shimura varieties = >;

(C) Critical values of zeta functions and periods of automorphic forms.

(B) includes (A) as 0-dimensional special case of canonical models. The relation of (B) and (C) is more involved, but (B) provides a solid foundation of the notion of the arithmetic automorphic forms. Also unitary Shimura varieties have recently attracted the interest (in particular by C. Skinner and E. Urban), see [2], in relation with the proof of the The Iwasawa Main Conjecture for $GL(2)$.

3.3 Integral Representations and Critical Values of the Zeta Functions

Automorphic forms are assumed scalar valued in this part. For Cases Sp and UT, Eisenstein series $E(z,s)$ associated to the maximal parabolic subgroup of G of Siegel type is introduced. Its analytic behaviour and those $\sigma \in 2^{-1}Z$ at which $E(z,\sigma)$ is nearly holomorphic and arithmetic are studied in [1]. This is achieved by proving a relation giving passage from *s* to *s* - 1 for $E(z,s)$, involving a differential operator, then examining Fourier coefficients of Eisenstein series using the theory of hypergeometric functions on tube domains.

For a Hecke eigenform f on G_A and an algebraic Hecke character χ on the idele group of K (in Case Sp, $K = F$), the zeta function $z(s, f, \chi)$ is defined. Regarding as an Euler product extended over prime ideals of F, the degree of the Eiler factor is $2n + 1$ in Case Sp, $4n$ in Case UT, and $2n$ in Case UB, except for finitely many prime ideals, see Chapter 5 of [1].

This zeta function is almost the same as the so called standard *L*-function attached to f twisted by χ but it turns out to be more general in the unitary case, see also [4].

Main results on critical values of Shimura's zeta functions is stated in Theorem 28.5, 28.8 (Cases Sp, UT), and in Theorem 29.5 in Case UB.

Theorem 3.1 (algebraicity of critical values in Cases Sp and UT):

Let $f \in v(\overline{Q})$ be a non zero arithmetical automorphic form of type Sp or UT.

Let χ be a Hecke character of K such that $\chi_a(x) = x_a^l \mid x_a \mid^{-l}$ `with $l \in Z^a$, and let $\sigma \in 2^{-1}Z$. Assume the following conditions (in the notations of Chapter 7 of [1] for the weight k_v, μ_v, l_v)

Case Sp

$2n+1-k_v-\mu_v \le k_v-\mu_v$, *where* $\mu_v = 0$
if $[k_v]-l_v \in 2Z$
and $\mu_v = 1$ *if* $[k_v]-l_v \notin 2Z$; $\sigma_0 - k_v + \mu_v$
for every $v \in a$ *if* $\sigma_0 > n$ *and*
$\sigma_0 - 1 - k_v + \mu_v \in 2Z$ *for every* $v \in a$
if $\sigma_0 \le n$.

Case UT

$4n-(2k_{vp}+l_v) \le 2\sigma_0 \le m_v - [k_v - k_{vp} - l_v]$
and $2\sigma_0 - l_v \in 2Z$ *for every* $v \in a$.

Further exclude the following cases:

(A) Case Sp $\sigma_0 = n+1, F = Q$ *and* $\chi^2 = 1$;

(B) Case Sp

$\sigma_0 = n + (3/2), F = Q;\ \chi^2 = 1$ *and* $[k] - l \in 2Z$;

(C) Case Sp $\sigma_0 = 0, c = g$ *and* $\chi = 1$;

(D) Case Sp

$0 < \sigma_0 \le= n, c = g, \chi^2 = 1$ *and* $[k] - l \in 2Z$;

(E) Case UT

$2\sigma_0 = 2n+1, F = Q, \chi_1 = \theta$ *and* $k_v - k_{vp} = l_v$;

(F) Case UT

$0 < 2\sigma_0 < 2n, c = g, \chi_1 = \theta^{2\sigma}$ *and*
the conductor of χ *is* τ

Then

$$z(\sigma_0, f, \chi)/\langle f, f\rangle \in \overline{Q} ,$$

where $d = [F:Q], |m| = \sum_{v\in a} m_v$, *and*

$$\varepsilon = \begin{cases} (n+1)\sigma_0 - n^2 - n, & Case\ Sp, k \in Z^a, and \sigma_0 > n_0, \\ n\sigma_0 - n^2, & Case\ Sp, k \notin Z^a, or \sigma_0 \prec n_0, \\ 2n\sigma_0 - 2n^2 + n, & Case\ UT \end{cases}$$

We establish a *p*-adic analogue of Theorem 28.8 (in Cases Sp and UT) representing algebraic parts of critical values as values of certain *p*-adic analytic zeta functions.

4 Constructing *P*-adic Zeta-functios via Quasi-modular Forms

We present here a new method of constructing *p*-adic zeta-functios based on the use of general quasimodular forms on classical groups.

The combinatorial structure of the Fourier coefficients of the holomorphic forms used in these constructions is quite complicated. We present a method of simpli_cation using nearly-holomorphic and general quasimodular forms, related to algebraic automorphic forms. It gives a new method of constructing *p*-adic zeta-functios using general quasi-modular forms and their Fourier coefficients. The symmetric space

$$H = G(R)/(\text{maximal-compact subgroup}\ \ K \times Center)$$

parametrizes certain families of abelian varieties $A_z (z \in H)$ so that $F \subset End(A_z) \otimes Q$. The CM-points z correspond to a maximal multiplication ring $End\ (A_z)$.

4.1 For the Group GL(2), N. Katz used Arithmetical Elements (Real-analytic and P-adic)

Instead of holomorphic forms in these representation spaces, these elements correspond also to quasimodular forms coming from derivatives which can be defined in general using Shimura's arithmeticity and the Maass-Shimura operators. A relation real-analytic $\leftrightarrow$ *p*-adic modular forms comes from the notion of *p*-adic modular forms invented by J.-P.Serre [5] as *p*-adic limits of *q*-expansions of modular forms with rational coefficients for $\Gamma = SL_2(Z)$. The present method of constructing *p*-adic automorphic *L*-functions uses general quasimodular forms, and their link to algebraic *p*-adic modular forms.

4.2 Real-analytic and P-adic Modular Forms

In Serre's case for $\Gamma = SL_2(Z)$, the ring M_p of *p*-adic modular forms contains $M = \oplus_{k\geq 0} M_k(\Gamma, Z) = Z[E_4, E_6]$, and it contains $E_2 = 1 - 24\sum_{n\geq 1}\sigma_1(n)q^n$.

On the other hand,

$$\tilde{E}_2 = -\frac{3}{\pi y} + E_2 = -12S + E_2, \text{ where } S = \frac{1}{4\pi y},$$

is a nearly holomorphic modular form (its coefficients are polynomials of *S* over *Q*). Let *N* be the ring of such forms. Then $\tilde{E}_2|_{S=0} = E_2$ and it was proved by J.-P. Serre that E_2 is a *p*-adic modular form. Elements of the ring $QM = N|_{S=0} = 0$ will be called general quasimodular forms. These phenomena are quite general and can be used in computations and proofs. In June 2014 in a talk in Grenoble, S.Boecherer extended these results to the Siegel modular case.

4.3 Using Algebraic and P-adic Modular Forms

There are several methods to compute various *L*-values starting from the constant term of the Eisenstein series in [5],

$$G_k(z) = \frac{\xi(1-k)}{2} + \sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n = \frac{\Gamma(k)}{(2\pi i)^k}\sum_{c,d}{}'(cz+d)^{-k};$$

and using Petersson products of nearly-holomorphic Siegel modular forms and arithmetical automorphic forms as in [1]: the Rankin-Selberg method, and the doubling method (pull-back method).

A known example is the standard zeta function $D(s, f, \chi)$ of a Siegel cusp eigenform $f \in S_n^k(\Gamma)$ of genus *n* (with local factors of degree 2n + 1) and χ a Dirichlet character.

Theorem (the case of even genus n ([6]), via the Rankin-Selberg method) gives a *p*-adic interpolation of the normailzed critical values $D^*(s, f, \chi)$ using Andrianov-Kalinin integral representation of these values $1+n-k \leq s \leq k-n$ through the Petersson product $\left\langle f, \theta_{T_0}\delta^r E\right\rangle$ where δ^r is a certain composition of Maass-Shimura di_erential operators, θ_{T_0} a theta-series of weight $n/2$, attached to a fixed $n\times n$ matrix T_0.

Theorem 4.1 (*p*-adic interpolation of $D(s, f, \chi)$)

(1) The case of odd genus (Boecherer-Schmidt, [7]): Assume that *n* is arbitrary genus, and a prime *p* ordinary then there exists a *p*-adic interpolation of $D(s, f, \chi)$

(2) Anh-Tuan Do (non-ordinary case, PhD Thesis of March 2014), via the doubling method: Assume that *n* is arbitrary genus, and *p* an arbitrary prime not dividing level of *f* then there exists a *p*-adic interpolation of $D(s, f, \chi)$.

Proof uses the following Boecherer-Garrett-Shimura identity (a pull-back formula) which allows to compute the critical values through certain double Petersson product by integrating over $z \in H_n$ the identity:

$$\begin{aligned}&\Lambda(l+2s,\chi)D(l+2s-n, f,\chi)f\\&=\left\langle f(w), E^{2n}_{l,\nu,\chi,s}(diag[z,w])_w\right\rangle,\end{aligned}$$

Here $k = l+\nu, \nu\geq 0, \wedge(l+2s,\chi)$ is a product of special values of Dirichlet *L*-functions and Γ -functions, $E^{2n}_{l,\nu,\chi,s}$, a higher twist of a Siegel-Eisenstein series on (*z*, *w*) $(z,w)\in H_n\times H_n$ (see [8]).

A *p*-adic construction uses congruences for the *L*-values, expressed through the Fourier coefficients of the Siegel modular forms and nearly-modular forms.

In the present approach of computing the Petersson products and *L*-values, an injection of algebraic nearly holomorphic modular forms into *p*-adic modular forms is used.

4.4 Injecting Nearly-holomorphic Forms into P-aduc Modular Forms

A recent discovery by Takashi Ichikawa (Saga University), [9], J. reine angew. Math., [10] allows to inject nearly-holomorphic arithmetical (vector valued) Siegel modular forms into *p*-adic modular forms. Via the Fourier expansions, the image of this injection is represented by certain quasimodular holomorphic forms like $E_2 = 1 - 24\sum_{n\geq 1}\sigma_1(n)q^n$, with algebraic Fourier expansions. This description provides many advantages, both computational and theoretical, in the study of algebraic parts of Petersson products and *L*-values, which we would like to develop here. This work is related to a recent preprint by S. Boecherer and Shoyu Nagaoka (On *p*-adic properties of Siegel modular forms, arXiv:1305.0604 [math.NT]) where it is shown that Siegel modular forms of level $\Gamma_0(p^m)$ are *p*-adic modular forms. Moreover they show that derivatives of such Siegel modular forms are *p*-adic. Parts of these results are also valid for vector-valued modular forms.

4.5 Arithmetical Nearly-holomorphic Siegel Modular Forms

Nearly-holomorphic Siegel modular forms over a subfield *k* of *C* are certain C^d-valued smooth functions f of $Z = X + \sqrt{-1}Y \in H_n$ given by the following expression $f(Z) = \sum_T P_T(S)q^T$ where *T* runs through the set B_n of all half-integral semi-positive matricies, $S = (4\pi Y)^{-1}$ a symmetric matrix, $q^T = \exp(2\pi\sqrt{-1}tr(TZ))$, $P_T(S)$ are vectors of degree d whose entries are polynomials over *k* of the entries of *S*.

4.6 Review of the Algebraic Theory

Following [11], consider the columns $Z_1, Z_2, \ldots Z_n$ of $Z \in H_n$ and the Z-lattice L_Z in C^n generated by $\{E_1, E_2, \ldots E_n, Z_1, Z_2, \ldots Z_n\}$, where $E_1, E_2, \ldots E_n$ are the columns of the identity matrix *E*. The torus $A_Z = C^n / L_Z$ is an abelian variety, and there is an analytic family $A \to H_n$ whose fiber over the point *Z* is A_Z. Let us consider the quotient space $H_n\Gamma(N)$ of the Siegel upper half space H_n of degree n by the integral symplectic group

$$\Gamma(N) = \left\{ r = \begin{pmatrix} A_r & B_r \\ C_r & D_r \end{pmatrix} \middle| \begin{matrix} A_r \equiv D_r \equiv 1_n \\ B_r \equiv C_r \equiv 0_n \end{matrix} \right\}$$

If $N > 3, \Gamma(N)$ acts without fixed points on $A = A_n$ and the quotient is a smooth algebraic family $A_{n,N}$ of abelian varieties with level *N* structure over the quasi-projective variety $H_{n,N}(C) = H_n / \Gamma(N)$ defined over $Q(\varsigma_N)$, where ς_N is a primitive *N*-th root of 1. For positive integers *n* and *N*, $H_{n,N}$ is the moduli space classifying principally polarized abelian schemes of relative dimension *n* with a symplectic level *N* structure.

4.7 De Rham and Hodge Vector Bundles

The fiber varieties A and $A_{n,N}$ give rise to a series of vector bundles over H_n and $H_{n,N}$. Notations:

$H^1_{DR}(A/H_n)$ and $H^1_{DR}(A_{n,N}/H_{n,N})$ the relative algebraic De Rham cohomology bundles of dimension 2n over H_n and $H_{n,N}$ respectively. Their fibers at $Z \in H_n$ are $H^1 := Hom_C(L_Z \otimes C, C)$ generated by α_i, β_i:

$$\alpha_i(\sum_j \alpha_j E_j + b_j Z_j) = a_i, \beta_i(\sum_j \alpha_j E_j + b_j Z_j) = b_i,$$
$$(i = 1, \ldots, n)$$

H^1_∞ the C^∞ vector bundle associated to H^1_{DR} (over H_n and $H_{n,N}$). It splits as a direct sum $H^1_\infty = H^{1,0}_\infty \oplus H^{0,1}_\infty$ and induces the Hodge decomposition on the De Rham cohomology of each fiber.

The summand $\omega = H^{1,0}_\infty$ is the bundle of relative 1-forms for either A/H_n and $A_{n,N}/H_{n,N}$. Let us denote by $\pi : A_{n,N} \to H_{n,N}$ the universal abelian scheme with 0-section s, and by the Hodge bundle of rank n defined as

$$E = \pi_*(\Omega^1_{A_{n,N}/H_{n,N}}) = s * H_{n,N}(\Omega^1_{A_{n,N}/H_{n,N}})$$

The bundle of holomorphic 1-forms on the base H_n or on $H_{n,N}$, is denoted Ω.

4.8 Algebraic Siegel Modular Forms

They are defined as global sections of E_ρ, the locally free sheaf on $H_{n,N} \otimes R$ obtained from twisting the Hodge bundle *E* by ρ.

Definition: Let *R* be a $Z[1/N, \varsigma_N]$-algebra. For an algebra homomorphism $\rho : GL_n \to GL_d$ over *R*, define algebraic Siegel modular forms over *R* as elements of $M_\rho(R) = H_0(H_{n,N} \otimes R, IE_\rho)$, called of weight ρ, degree *n*, level *N*. If $\rho = \det^{\otimes k} : GL_n \to G_m H_0$ called of weight ρ, then elements of $M_k(R) = M_{\det^k}(R)$ are called of weight *k*.

For $R = C$, each $Z \in H_n$, let $A_Z = C^n/(Z^n + C^n \cdot Z)$ be the corresponding abelian variety over *C*, and $u_{1,} ..., u_n$ be the natural coordinates on the universal cover C^n of A_Z. Then *E* is trivialized over H_n by $du_{1,} ..., du_n$, and $f \in M_\rho(C)$ is a complex analytic section of E_ρ on $H_{n,N}(C) = H_n / \Gamma(N)$.

Hence, an element $f \in M_\rho(C)$ is a C^d-valued holomorphic function on H_n satisfying the ρ-automorphic condition:

$$f(Z) = \rho(C_r Z + D_r)^{-1} \cdot f(\gamma(Z))$$

$$\left(Z \in H_{n,r} = \begin{pmatrix} A_r & B_r \\ C_r & D_r \end{pmatrix} \right)$$

because $A_Z \xrightarrow{\sim} A_{rZ} : {}^t(u_{1,} ..., u_n) \mapsto (CZ + D)^{-1\,t}(u_{1,} ..., u_n)$ and *r* acts equivariantly on the trivialization of *E* over H_n as the left multiplication by $(CZ + D)^{-1}$.

4.9 Algebraic Fourier Expansion

It can be defined algebraically using an algebraic test object over the ring $R_n = Z[q_{11}, ..., q_{nn}][q_{ij}, q_{ij}^{\pm 1}]_{i,j=1,...,n}$ are variables with symmetry $q_{ij} = q_{ji}$. Mumford constructs an object represented over *n* as

$R_{n,N} \otimes Z[1/N, \varsigma_N]M)^d (G_m)^n / \langle (q_{ij})_{i=1,...,n} \mid 1 \le j \le n \rangle$,

$(G_m)^n = Spec(Z[x_1^{\pm 1}, ..., x_n^{\pm 1}])$.

(D. Mumford, An analytic construction of degenerating abelian, varieties over complete rings, Compositio Math 24 (1972) 239-272).

For the level *N*, at each 0-dimensional cusp *c* on $H^*_{n,N}$ (Satake's minimal compacti_cation of $H_{n,N}$), this construction gives an abelian variety over the formal power series ring

$$R_{n,N} = Z[1/N, \varsigma_N][q_{11}^{1/N}, ..., q_{nn}^{1/N}][q_{ij}^{\pm 1}]_{i,j=1,...,n}$$

with a symplectic level N structure, and $\omega_i = dx_i / x_i (1 \le i \le n)$ form a basis of regular 1-forms. We may view algebraically Siegel modular forms as certain sections of vector bundles over $H_{n,N}$. Using the morphism $Spec(R_{n,N}) \to H_{n,N}, E$ becomes $(R_{n,N} \otimes R)^n$ in the basis $\omega_i = dx_i / x_i (1 \le i \le n)$ of regular 1-forms.

4.10 Fourier Expansion Map and Q-expansion Principle

For an algebraic representation $\rho : GL_n \to GL_d$, E_ρ becomes in the above basis ω_i

$$E_\rho \times H_{n,N \otimes R} Spec(R_{n,N} \otimes R) = (R_{n,N} \otimes R)^d.$$

For an R-module M, the space of Siegel modular forms with coefficients in M of weight ρ is defined as

$$M_\rho(M) = H^0(H_{n,N\otimes R}, E_\rho \otimes_R M).$$

Then the evaluation on Mumford's abelian scheme gives a homomorphism

$$F_c : M_\rho(M) \to R_{n,N} \otimes Z[1/N, \varsigma_N]M)^d$$

which is called the Fourier expansion map associated with c. According to [10], Theorem 2, F_c satisfies the following q-expansion principle:

If M' is a sub R-module of M and $f \in M_\rho(M)$ satisfies that $F_c(f) \in R_{n,N} \otimes Z[1/N, \varsigma_N]M')^d$, then $f \in M_\rho(M')$.

For q-expansion principle in the unitary case can be seen in [4] and [12].

4.11 Algebraic nearly Holomorphic Forms as Formal Fourier Expansions over a Commutative Ring A

Algebraically we use the notation

$$q^T = \prod_{i=1}^{n} q_{ij}^{T_{ii}} \prod_{i<j} q_{ij}^{2T_{ii}} \in A[q_{11},...,q_{nn}][q_{ij}, q_{ij}^{\pm 1}]_{i,j=1,...,n}$$

(with $q^T = \exp(2\pi i tr(TZ)), q_{ij} = \exp(2\pi(\sqrt{-1}Z_{ij}))$ for A = C). The elements q^T form a multiplicative semi-group so that $q^{T_1 \cdot T_2} = q^{T_1+T_2}$, and one may consider f as a formal q-expansion over an arbitrary ring A via elements of the semi-group algebra $A[q^{B_n}]$.

Algebraic definition of arithmetical nearly holomorphic forms, see [1] $f \in S_e(Sym^2(A^n), A[q^{B_n}]^d$, where S_e denotes the A-polynomial mappings of degree e on symmetric matricies $S \in Sym^2(A^n)$ of order n with vector values in $A[q^{B_n}]^d$.

Notation: $f = \sum_T a_T(S)q^T \in N(A)$.

General quasimodular forms. For all $\kappa(f) = \sum_T a_T(0)q^T = f \mid s = 0$ define general quasimodular forms as elements of the form

Notation: $\kappa(f) \in QM(A)$.

4.12 Computing the Petersson Products

The Petersson product of a given modular form $f(Z) = \sum_T aTq^T \in M_\rho(\overline{Q})$ by another modular form $h(Z) = \sum_T aTq^T \in M_{\rho*}(\overline{Q})$ produces a linear form

$$l_f :\mapsto \frac{\langle f,h\rangle}{\langle f,f\rangle}$$

defined over a subfield $K \subset Q$. Thus ℓ_f can be expressed through the Fourier coefficients of h in the case when there is a finite basis of the dual space consisting of certain Fourier coefficients: $\ell_{T_i} : h \mapsto b_{T_i}(i = 1,...,n)$. It follows that $\ell_f(h) = \sum_i irb_{T_i}$ where $r_i \in k$

4.13 Applications to Constructions of P-adic L-functions

There exist two kinds of L-functions:

Complex L-functions (Euler products) on $C = Hom(R_+^*; C^*)$ and p-adic L-functions on the C_p-analytic group $Hom_{cont}(Z_p^*; C_p^*)$ (Mellin transforms L_μ of p-adic measures μ on Z_p^*).

Both are used in order to obtain a number (L-value) from an automorphic form. Such a number can be algebraic (after normalization) via the embeddings,

$\overline{Q} \to C, \overline{Q} \to C_p = \hat{\overline{Q}}_p$ and we may compare the complex and p-adic L-values at many points.

How to define and to compute p-adic L-functions? The Mellin transform of a p-adic distribution μ on Z_p^* gives an analytic function on the group of p-adic characters

$$x \mapsto L_\mu(x) = \int_{Z_p^*} x(y)d\mu(y), x \in X_{Z_p^*} = Hom_{cont}(Z_p^*; C_p^*)$$

A general idea is to construct p-adic measures directly from Fourier coefficients of modular forms proving Kummer-type congruences for L-values. Here we present a new method to construct p-adic L-functions via quasimodulat forms.

4.14 How to Prove Kummer-type Congruences using the Fourier Coefficients?

Suppose that we are given some L-function $L^*(s,\chi)$ attached to a Siegel modular form f and assume that for infinitely many "critical pairs" $s_j,\chi_j)$ one has an integral representation $L^*(s,\chi)=\langle f,h_j\rangle$ with all $h_j=\sum_T b_{j,T}q^T\in M$ in a certain finite-dimensional space M containing f and defined over $\overline{Q}$. We want to prove the following Kummer-type congruences:

$$\forall_x\in Z_p^*,\sum_j\beta_j\chi_j x^{k_j}\equiv 0 \quad \text{mod} \quad p^N\Rightarrow\sum_j\beta_j\frac{L^*f(s,\chi)}{\langle f,f\rangle}\equiv 0$$

mod p^N for any choice of

$$\beta_j\in\overline{Q},k_j=\begin{cases}s_j-s_0 & if\ s_0=\min_j s_j\ or\\ k_j=s_0-k_j & if\ s_0=\max_j s_j.\end{cases}$$

Using the above expression for $\ell_f(h_j)=\sum_i ri,jb_{j,T_i}$, the above congruences reduce to

$$\sum_{i,j}r_{i,j}\beta_j b_{j,T_i}\equiv 0 \quad \text{mod} \quad p^N.$$

4.15 Reduction to a finite Dimensional Case

In order to prove the congruences

$$\sum_{i,j}r_{i,j}\beta_j b_{j,T_i}\equiv 0 \quad \text{mod} \quad p^N.$$

in general we use the functions h_j which belong only to a certain infinite dimensional $\overline{Q}$-vector space

$$M=M(\overline{Q})$$

$$M(\overline{Q}):=\bigcup_{m\geq 0}M_{\rho*}(\overline{Q})(Np^m,\overline{Q}).$$

Starting from the functions h_j, we use their caracteristic projection $\pi=\pi^\alpha$ on the characteristic subspace M^α (of eigenvectors) associated to a non-zero eigenvalue α of Atkin's U-operator on f which turns out to be of fixed finite dimension so that for all $j,\pi^\alpha(h_j)\in M^\alpha$.

4.16 From Holomorphic to nearly Holomorphic and P-adic Modular Forms

Next we explain how to treat the functions h_j which belong to a certain infinite dimensional Q-vector space $N\subset N_\rho(\overline{Q})$ (of nearly holomorphic modular forms).

Usually, h_j can be expressed through the functions $\delta_{kj}(\varphi_0(\chi_j))$ for a certain non-negative power k_j of the Maass-Shimura-type differential operator applied to a holomorphic form $\varphi_0(\chi_j)$. Then the idea is to proceed in two steps:

(1) To pass from the infinite dimensional $\overline{Q}$-vector space $N=N(Q)$ of nearly holomorphic modular forms,

$$N(Q):=\bigcup_{m\geq 0}N_{k,r}(Q)(Np^m,Q)$$

(of the depth r): to a fixed finite dimensional characteristic subspace $N^\alpha\subset N(N_\rho)$ of U_p in the same way as for the holomorphic forms. This step controls Petersson products using conjugate f^0 of an eigenfunction f_0 of $U(p)$:

$$\langle f^0,h\rangle=\alpha^{-m}\langle f^0,h\,|\,U(p)^m\rangle=\langle f^0,\pi^\alpha(h)\rangle.$$

(2) To apply Ichikawa's mapping $\iota_p:N(N_p)\to M_p(N_p)$ to a certain space $M_p(N_p)$ of p-adic Siegel modular forms. Assume algebraically,

$$h_j=\sum_T b_{j,T}(S)q^T\mapsto(h_j)=\sum_T b_{j,T}(0)q^T$$

which is also a certain Siegel quasi-modular form. Under this mapping, computation become much easier, as the action of δ^{kj} becomes simply a k_j-power of the Ramanujan Θ-operator

$$\Theta:\sum_T b_T q^T\mapsto\sum_T\det(T)b_T(0)q^T$$

in the scalar-valued case. In the vector-valued case such operators were studied in preprint by S. Boecherer and Shoyu Nagaoka (On p-adic properties of Siegel modular forms, arXiv:1305.0604 [math.NT]).

After this step, proving the Kummer-type congruences reduces to those for the Fourier coefficients the quasimodular forms $\kappa(h_j(\chi_j))$ which can be explicitely evaluated using the Θ-operator.

4.17 Computing with Siegel Modular Forms over a Ring A

There are several types of Siegel modular forms (vector-valued, nearly-holomorphic, quasi-modular, *p*-adic). Consider modular forms over a ring $A = C, C_p, \Lambda = Z_p[T], \ldots$ as certain formal Fourier expansions over *A*. Let us fix the congruence subgroup Γ of a nearly holomorphic modular form $f \in N_p$ and its depth *r* as the maximal *S*-degree of the poynomial Fourier Fourier coefficients $a_T(S)$ of a nearly holomorphic form

$$h_j = \sum_T b_{j,T}(S)q^T \mapsto k(h_j) = \sum_T b_{j,T}(0)q^T$$

which is also a certain Siegel quasi-modular form. Under this mapping, computation become much easier, as the action of δ^{kj} becomes simply a k_j-power of the Ramanujan Θ-operator

$$\Theta: \sum_T a_T(S)q^T \in N(A),$$

Over *R*, and denote by $N_{\rho,r}(\Gamma, A)$ the *A*-module of all such forms. This module is often locally-free of finite rank, that is, it becomes a finite-dimensional *F*-vector space over the fraction field $F = F_{rac}(A)$.

4.18 Types of Modular Forms

(1) M_ρ (holomorphic vector-valued Siegel modular forms attached to an algebraic representation $\rho: GL_n \to GL_d$);

(2) $QM = N|_{S=0}$ (quasi-modular vector-valued forms attached to ρ);

(3) N_ρ (holomorphic vector-valued Siegel modular forms, algebraic *p*-adic vector-valued forms attached to ρ over a number field $k \subset \overline{Q} \to C_p$).

Definitions and interrelations:

(1) $QM_{\rho,r} = k(N_{\rho,r})$ where $\kappa: f \mapsto f|_{S=0} = \sum_T P_T(0)q^T$ with the notation $R_{n,\infty} = C[[q_{11},\ldots,q_{nn}]][q_{ij}, q_{ij}^{-1}]_{i,j=1,\ldots,n}$

(2) $: M^b_{\rho,r}(R,\Gamma))) = F_c(\iota_p(N_{\rho,r}(R,\Gamma))) \subset R^d_{n,p}$

where $R_{n,p} = C_p[[q_{11},\ldots,q_{nn}]][q_{ij}, q_{ij}^{-1}]_{i,j=1,\ldots,n}$.

Let us fix the level Γ, the depth r, and a subring R of $\overline{Q}$, then all the *R*-modules $M_{\rho,r}(R,\Gamma))), N_{\rho,r}(R,\Gamma)))QM_{\rho,r}(R,\Gamma))), M^b_{\rho,r}(R,\Gamma)))$ are then locally free of finite rank.

In interesting cases, there is an inclusion

$$QM_{\rho,r}(R,\Gamma) \to M^b_{\rho,r}(R,\Gamma).$$

If $\Gamma = SL_2(Z), k = 2, P = E_2$ is a *p*-adic modular form, see [5], p.211.

Question: Prove it in general! (after discussions with S.Boecherer and T. Ichikawa)[1].

5 Applications to Families of Arithmetical Automorphic Forms

We treat only the Siegel modular case here but the results can be extended to the general Sp- and unitary cases (UT in Shimura's terminology).

5.1 Computing with Families of Siegel Modular Forms

Let $\Lambda = Z_p[T]$ be the Iwasawa algebra, and consider Serre's ring

$$R_{n,\wedge} = \Lambda[[q_{11},\ldots,q_{nn}]][q_{ij}, q_{ij}^{-1}]_{ij=1,\ldots,n}$$

For any pair (k, χ) as above consider the homomorphisms:

$$\kappa_{\kappa,\chi}: \Lambda C_p \mapsto R_{n,\Lambda} \mapsto R_{n,C_p}$$

where $T \mapsto \chi(1+p)(1+p)^k - 1$.

Definition 5.1 (Families of Siegel modular forms) Let $f \in R_{n,\Lambda}$ such that for infinitely many pairs (k, χ) as above,

$$\kappa_{k,\chi}(f) \in M_{\rho k}((i_p(\overline{Q})) \overset{F_c}{\mapsto} R^d_{n,C_p}$$

[1] In June 2014, an affirmative answer is given by S.Boecherer for the Siegel modular group.

is the Fourier expansion at c of a Siegel modular form over $\overline{Q}$. All such f generate the Λ-submodule $M_{\rho k}(\Lambda) \subset R^{d}_{n,\Lambda}$ of Λ-adic Siegel modular forms of weight ρ.

In the same way, the Λ-submodule

$QM_{\rho k}(\wedge) \subset R^{d}_{n,\wedge}$

of Λ-adic Siegel quasi-modular forms is defined.

5.2 Examples of Families of Siegel Modular Forms

It can be constructed via differential operators of Maass

$$\Delta = \det\left(\frac{1+\delta_{ij}}{2}\frac{\partial}{\partial_{z_{ij}}}\right),$$

so that $\Delta q^T = \det(T)q^T$. Shimura's operator

$\delta_k f(Z) =$

$$(-4\pi)^{-n}\det(Z-\overline{Z})^{\frac{1+n}{2}-k}\Delta(\det(Z-\overline{Z})^{k-\frac{1+n}{2}+1} f)(Z)$$

acts on q^T using $\rho_r : GL_n(C) \to GL(\wedge^r C^n)$ and its adjoint ρ_r^*:

$$\delta_k(q^T) = \sum_{l=0}^{n} c_{n-l}(k+1-\frac{1+n}{2}) tr({}^t\rho_{n-l}(S)\rho_l^*(T))q^T,$$

where

$$c_{n-l}(s) = s(s-\frac{1}{2})...(s-\frac{n-l-1}{2}), S = (2\pi i(\overline{z}-z))^{-1}.$$

Nearly holomorphic $\wedge$-adic Siegel-Eisenstein series as in [14] can be produced from the pairs $(-s,\chi)$: if s is a nonpositive integer such that

$k+2s > n+1$

$$E_k(Z,s,\chi) = \prod_{i=0}^{-s-1} c_n(k+2s+2i)^{-1}\delta_{k+2s}^{(-s)}(E_{k+2s}(Z,0,\chi)).$$

Ichikawa's construction: quasi-holomorphic (and p-adic) Siegel-Eisenstein series obtained in [10] using the injection ι_p:

$$\iota_p(\pi^{ns}E_k(Z,s,\chi)) = \prod_{i=0}^{-s-1} c_n(k+2s+2i)^{-1}\sum_T \det(T)^{-s} b_{k+2s}(T)q^T,$$

where

$$E_{k+2s}(Z,0,\chi) = \sum\nolimits_T b_{k+2s}(T)q^T, k+2s > n+1, s \in Z.$$

A two-variable family is for the parameters $(k+2s,s), k+2s > n+1, s \in Z$ will be now constructed. Normalized Siegel-Eisenstein series of two variables Let us start with an explicit family described in [13], [14], [15] as follows

$$\varepsilon_k^n = E_k^n(z)2^{n/2}\varsigma(1-k)\prod\nolimits_{i=1}^{[n/2]}\varsigma(1-2k+2i) = \sum\nolimits_T a_T(\varepsilon_k^n)q^T,$$

where for any non-degenerate matrice T of quadratic character ψ_T.

Proposition 5.2 Let $k > m+1$.

(1) For any non-degenerate matrix $h \in C_m$ the following equality holds

$$a_h(\varepsilon_k^m) = 2^{-\frac{m}{2}}\det h^{k-\frac{m+1}{2}} H_h(k)$$
$$\times\begin{cases} L(1-k+\frac{m}{2},\psi_h)C_h^{\frac{m}{2}-k+(1/2)}, m \quad even, \\ 1, \qquad m \quad odd, \end{cases}$$

where C_h is the conductor of ψ_h.

(2) for any prime $p > 2$, and $\det(2h)$ not divisible by p, define the p-regular part $a_h(\varepsilon_k^m)^{(p)}$ of the coefficient $a_h(\varepsilon_k^m)$ of ε_k^m by introducing the factor

$$a_h(\varepsilon_k^m) = 2^{-\frac{m}{2}}\det h^{k-\frac{m+1}{2}} H_h(k)$$
$$\times\begin{cases} (1-\psi_h(p)p^{k-\frac{m}{2}-1})C_h^{\frac{m}{2}-k+(1/2)}, m \quad even, \\ 1, \qquad m \quad odd, \end{cases}$$

Then $a_h(\varepsilon_k^m)^{(p)}$ is a p-adic analytic Iwasawa function of $t = (1+p)^k - 1$ for all k with ω^k fixed, and divided by the elementary factor $1-\psi_h(c_h)c_h^{k-\frac{m}{2}}$.

Then Ichikawa's construction is applicable and it provides a two-variable family.

5.3 Further Examples of Families of Siegel Modular Forms

Ikeda-type families of cusp forms of even genus [16]. Start from a p-adic family

$$\varphi = \{\varphi_{2k}\} : 2k \mapsto \varphi_{2k} = \sum_{n=1}^{\infty} a_n(2k)q^n \in Q[q] \subset C_p[q]$$

where the Fourier coefficients $a_n(2k)$ of the normalized cusp Hecke eigenform φ_{2k} and one of the Satake p-parameters $\alpha(2k):\alpha_p(2k)$ are given by certain p-adic analytic functions $k \mapsto a_n(2k)$ for $(n,p)=1$.

The Fourier expansions of the modular forms $F = F_{2n}(\varphi_{2k})$ can be explicitly evaluated where

$$L(F_{2n}(\varphi), St, s) = \varsigma(s)\prod_{i=1}^{2n} L(\varphi, s+k+n-i).$$

This sequence provide an example of a p-adic family of Siegel modular forms.

Ikeda-Myawaki-type families of cusp forms of $n=3$, [16].

Families of Klingen-Eisenstein series extended from $n=2$ to a general case (reported in Journées Arithmétiques, Grenoble, July 2013).

6 Families of Klingen-Eisenstein Series

6.1 Geometric Constructions of Distributions for Klingen-Eisenstein Series

They will be obtained staring from those for Siegel-Eisenstein series at any prime *p*. Next, axioms for *p*-adic doubling method will be stated.

Let us describe distributions for Klingen-Eisenstein series and Siegel-Eisenstein series.

Let *p* be a prime, Γ_n the Siegel modular group of degree *n*.

Fix a Siegel cusp eigenform $f = \sum_T a_T q^T \in S_k^n(\Gamma^n)$, where T runs through half-integral positive symmetric matrices.

6.2 Key Ingredients of Our Construction

Siegel-Eisenstein distributions generalizing those in [17]

$$E_{k,b,p^v}^{m+n} = \sum_{(c,d)} \det(cz+d)^{-k}$$

$$\det c \equiv b \bmod p^v$$

These are functions on the Siegel upper half plane $H_{m+n} = \{z={}^t z \in M_{m+n}(C) \mid \operatorname{Im}(z) > 0\}$ of degree $m+n, (c,d)$ runs over equivalence classes of all coprime symmetric couples, that is $r = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ runs over equivalence classes

of $\Gamma = \Gamma^{m+n}$ modulo the Siegel parabolic $P = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$;

Higher twists $E_{m,n}(\chi_1,\chi_2)$ of Siegel -Eisenstein distributions constructed as in [18]. These are certain functions on $(Z,\tau) \in H_m \times H_n$;

Pull-back formula

$$\left\langle E_{m,n}(\chi_1,\chi_2)(Z,\tau), f(\tau)\right\rangle_\tau = C_{\chi_1,\chi_2,m,n} L(f,\chi_{1,2},k-n) Em(f,\chi_1,\chi_2)$$

representing the Klingen-Eisenstein series $E_m(f,\chi_1,\chi_2)$ on $Z \in H_m (m \ge n)$, where $L(f,\chi_{1,2},s)$ denotes the standard *L*-function of *f* with a certain Dirichlet character $\chi_{1,2}$ attached to f,χ_1,χ_2.

Canonical projection of the Klingen-Eisenstein distributions

$$\Phi = \left\langle E_{m,n}(\chi_1,\chi_2), f^0\right\rangle_\tau$$

onto a fixed-level finite-dimensional subspace using the $U_0(p)$-operator of degree *n* and a non-zero eigenvalue $\alpha_0(p)$ of $U_0(p)$, as in [19] and in [20].

6.3 Axioms for P-adic Doubling Method

Here we present an axiomatic proceedure which can be used in a *p*-adic doubling method.

We generalize passage from Siegel-Eisenstein series to Klingen-Eisenstein series on the level distributions with values in modular forms (modular distributions,

usinf the ideas in [19], [20],... Such a distrinution Φ on Z_p^* is characterized by all their integrals $\Phi(\chi)$

over Dirichlet characters χ mod p^r, which define families of modular forms

$$\chi \mapsto \Phi(\chi).$$

Starting from a family of distributions with values in functions on H_{2n} one can obtain distributions with values in functions on H_n by taking a partial Petersson product. We analyze conditions to obtain bounded distributions in this way. Thus we start from a family of functions

$$F_\chi : H_{2n} \to C$$

where χ runs through primitive characters χ mod p^v and consider exterior twist

$$\underbrace{\sum_{X \in Z^{m,n} \bmod p^v} F_\chi \begin{pmatrix} z_1 & \frac{X}{N} \\ \frac{X^t}{N} & z_4 \end{pmatrix}}_{g_x(z_1,z_4)}$$

which will be of Haupttypus of level p^{2v}.

Next let us fix two cusp forms f_1, f_2 and consider

$$\chi \mapsto \int_{\Gamma_0(p^{2v})\backslash H} \int f_1^0, f_2^0 \overline{g_\chi(z_1, z_4)} dz_1 dz_4 \in C$$

to get p-adic interpolation and use U_0^{2v} (in z_1 and z_4) to get finite dimensional space.

6.4 Proving Congruences using Fourier Expansion

Using the notation

$$F_\chi = \sum_T a(T,\chi)\exp(2\pi i tr(TZ))$$

we describe congruences satisfied by the family of functions obtained by taking the Petersson products of with a fixed form. We use the action of $U(p)$ on both variables, then

$$g_\chi(z_1,z_4)|^{z_1} U(p^j)|^{z_1\,4} U(p^j)$$
$$= \sum_{T_1,T_4} c_j(T_1,T_4,\chi)\exp(2\pi i tr(T_1 Z_1 + T_4 Z_4)),$$
$$c_j(T_1,T_4,\chi) = \sum_{\chi_4} a\left(\begin{pmatrix} p^j T_1 & T_2 \\ T_2 N & p^j T_4 \end{pmatrix}, \chi\right)$$
$$r \geq 1$$

$$c_j(T_1,T_4,\chi)$$
$$= \frac{1}{p^r} \sum_{\substack{\chi \\ conductor(\chi) \leq p^r}} a\left(\begin{pmatrix} p^{n-r}T_1 & T_2 \\ T_2 N & p^{n-r}T_4 \end{pmatrix}, \chi\right)\chi(\det T_2)$$

6.5 Eisenstein Series and Theta Series

The weighted sum F_χ of Fourier coefficients must satisfy certain p-adic congruences in order to interpolate the values of the integrals

$$\alpha_{f_1}^{-2v}\alpha_{f_2}^{-2v}\int_{\Gamma_0(p^{2v})\backslash H} \int f_1^0, f_2^0 \overline{g_\chi(z_1,z_4)} dz_1 dz_4 \in C$$

Questions about other families: Ikeda lifts What should be F_χ for the Ikeda lift? (see [13], [21]) Such families should imitate the variations

$$\chi \mapsto \sum_{C,D} \underbrace{\chi(\det C)}_{this\ is\ variation} \det(CZ+D)^{-k}$$

Theta series Another natural example of such a family in double variables is produced by the theta series, one is reduced to the exponential sum

$$\Theta_S \mapsto \Theta_{S,\chi} = \underbrace{\sum_X \chi(\det C)\exp(2\pi i tr(S[X]Z))}_{a\ possibility}$$

6.6 Axioms for P-adic Pull Back Formula

Here let us start with a family of functions

$$F_\chi : H_{m+n} \to C$$

where χ runs through primitive characters χ mod p^v and consider exterior twist

$$\underbrace{\sum_{X \in Z^{m,n} \bmod p^v} F_\chi \begin{pmatrix} z_1 & \frac{X}{N} \\ \frac{X^t}{N} & z_4 \end{pmatrix}}_{g_x(z_1,z_4)}$$

which will be of Haupttypus of level p^{2v} of $n+m$.

Next let us fix two cusp forms f_1, f_2 and consider

$$\chi \mapsto \int_{\Gamma_0(p^{2v})\backslash H} \int f_1^0(z_1) f_2^0(z_4) \overline{g_\chi(z_1,z_4)} dz_1 dz_4 \in C$$

to get p-adic interpolation and use again U_0^{2v} (in z_1 and z_4) to get finite dimensional space.

Acknowledgement

Many thanks to Thomas Johnson, Editor of the Journal of Mathematical Sciences for soliciting a paper for this Issue of 2015;

The paper grew from several talks: for “Journées Arithmétiques” (University Grenoble-1, 2013), for the Workshop “Explicit Methods in the Theory of Automorphic Forms” (Shanghai, Tongji University, March 2014), Rencontres Clermont-Ferrand-Grenoble (Institut Fourier, June 2014), Workshop “Algebraic Geometry and Number Theory” (June 2014), Workshop “Zeta V” (Laboratary J.-V. Poncelet, Moscow, December 2014), Workshop “Non-Archiledean Analysis of VIASM, Halong Bay”, and numerous talks in the Universities of Vinh, Dong Hoi, Thanh Long University, Institut of Technology Hoc Vien Ki Thuat Quan Su in Vietnam, April 2015. Let all these Institutions be thanked for their kind hospitality;

Author is also very grateful to L. Schwartz, M. Morales, and Yolande Jimenez (Format Vietnam) for organizing his travel to Hanoi. Author’s special thanks goes to Khoai Ha Huy for its invitation to VIASM, and to Siegfried Boecherer and Vladimir Berkovich for fruitful discussions.

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