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History-dependent Evolutionary Quasi-variational Inequalities with Viscosity and Volterra Integral Term

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In this paper, we study a class of history-dependent evolutionary quasi-variational inequalities with viscosity and Volterra integral term. We prove existence, uniqueness and convergence results for the weak solution, which is obtained by using arguments for variational inequalities of the second kind and Banach's fixed-point theorem.

Keywords: Banach's fixed point theorem, Gronwall inequality, Variational inequalities, Volterra integral term.

1. Introduction

We have investigated in [1, 2, 3] abstract evolutionary variational inequalities of the form:

$$\begin{aligned}&\left(Au(t), v-\dot{u}(t)\right)_V\\&+\left(\int_0^t \mathcal{R}\,(t-s)u(s)ds, v-\dot{u}(t)\right)_V\\&+j(v)-j\left(\dot{u}(t)\right)\geq\left(f(t), v-\dot{u}(t)\right)_V\\&\forall v\in V, t\in[0,T]\end{aligned}\tag{1.1a}$$

$$u(0)=u_0\tag{1.1b}$$

in a real Hilbert space V. In (1.1a)-(1.1b), $T>0$ and $[0,T]$ is the time interval of interest, $A: V\to V$ is a nonlinear operator, $\mathcal{R}(t): V\to V$ is a linear continuous operator for all $t\in[0,T]$, $j: V\to\mathbb{R}$ is a convex function and $f:[0,T]\to V$ is given function. Here and everywhere in this paper, the dot above a variable represents its derivative with respect to the time. Problems of this form arise in the study of quasistatic frictionaless contact models involving elastic materials with long memory [4, 5, 6, 7]. In such problems $u:[0,T]\to V$ represents the displacement field, A and $\mathcal{R}$ are the elasticity and relaxation operators, respectively. The function f is related to the given body forces and surface traction. The results in [1, 2, 3] deal with the existence of a unique solution to problem (1.1a)-(1.1b), which is obtained by using arguments for evolutionary variational inequalities and Banach's fixed-point theorem.

In this paper, we study a class of history-dependent evolutionary quasi-variational inequalities with viscosity and Volterra integral term for which the functional j depends on the integral of the solution [8]. To this end, let $(Y,\|\cdot\|_Y)$ be a normed space, assume that $j: Y\times V\to\mathbb{R}$, and consider an operator $\mathcal{S}: C([0,T];V)\to C([0,T];Y)$. We are interested in providing conditions that guarantee the unique solvability of the following problem

$$\begin{aligned}&\left(Au(t), v-\dot{u}(t)\right)_V\\&+\left(B\dot{u}(t), v-\dot{u}(t)\right)_V\\&+\left(\int_0^t \mathcal{R}\,(t-s)u(s)ds, v-\dot{u}(t)\right)_V\\&+j(\mathcal{S}(u), v)-j\left(\mathcal{S}(u),\dot{u}(t)\right)\\&\geq\left(f(t), v-\dot{u}(t)\right)_V\ \forall v\in V, t\in[0,T]\end{aligned}\tag{1.2a}$$

$$u(0)=u_0\tag{1.2b}$$

in which the unknown is the function $u:[0,T]\to V$ and $B: V\to V$ is continuous nonlinear operator.

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The rest of the paper is structured as follows. In Section 2, we prove the existence and uniqueness of the solution to problem (1.2a)-(1.2b). It is based on arguments for variational inequalities of the second kind and Banach's fixed-point theorem. In Section 3, we study the behavior of the solution with respect to perturbations of operators A, $\mathcal{R}$ and the functional j, and derive a convergence result.

2. Assumptions and Equivalent Problem

In this section, we list the assumptions on the data and we obtain an equivalent problem to problem (1.2a)-(1.2b).

We suppose in what follows that V is a real Hilbert space endowed with the inner product $(\cdot,\cdot)_V$ and the associated norm $\|\cdot\|_V$ We also denote by $\mathcal{L}(V)$ the space of linear and continuous operators from V to V with norm $\|\cdot\|_{\mathcal{L}(V)}$. We denote by $\mathcal{C}([0,T];V)$ the space of continuous functions from $[0,T]$ to V, with the norm

$$\|x\|_{\mathcal{C}([0,T];X)} = \max_{t\in[0,T]}\|x(t)\|_V$$

In the study of problem (1.2a)-(1.2b), we assume that: $B: V \to V$ is a strongly monotone Lipschitz continuous operator, i.e.,

$$\exists m > 0, (Bv_1 - Bv_2, v_1 - v_2)_V \geq m\|v_1 - v_2\|_V^2, \tag{2.1a}$$

$$\exists L > 0, \|Bv_1 - Bv_2\|_V \leq L\|v_1 - v_2\|_V. \tag{2.1b}$$

$A: V \to V$ is a Lipschitz continuous operator, i.e.

$$\exists L' > 0, \|Av_1 - Av_2\|_V \leq L'\|v_1 - v_2\|_V. \tag{2.2}$$

The relaxation operator $\mathcal{R}$ satisfy

$$\mathcal{R} \in \mathcal{C}\big([0,T];\mathcal{L}(V)\big). \tag{2.3}$$

The function $j: Y \times V \to \mathbb{R}$ satisfy

$$\forall y \in Y, j(y,\cdot): V \to \mathbb{R} \text{ is convex and l.s.c.}; \tag{2.4a}$$

$$\begin{cases} j(w_1,v_2) - j(w_1,v_1) \\ +j(w_2,v_1) - j(w_2,v_2) \\ \leq \alpha\|w_1 - w_2\|_Y\|v_1 - v_2\|_V \\ \forall(w_i,v_i) \in Y \times V, \alpha > 0. \end{cases} \tag{2.4b}$$

The operator $\mathcal{S}: \mathcal{C}([0,T];V) \to \mathcal{C}([0,T];Y)$ satisfies

$$\begin{cases} \exists c_0 > 0, \forall v_1, v_2 \in \mathcal{C}([0,T];V): \\ \|\mathcal{S}(v_1) - \mathcal{S}(v_2)\|_Y \\ \leq c_0\displaystyle\int_0^t \|v_1(s) - v_2(s)\|_V\, ds \end{cases} \tag{2.5}$$

and, finally, we assume that

$$f \in \mathcal{C}([0,T];V) \tag{2.6}$$

$$u_0 \in V \tag{2.7}$$

To study the problem (1.2a)-(1.2b), we will transform this problem to a simple equivalent problem as follows.

Define the functional $\varphi: V \times V \to \mathbb{R}$ by

$$\varphi(u,v) = (Au,v)_V + j(\mathcal{S}(u),v) + \left(\int_0^t \mathcal{R}(t-s)u(s)ds, v\right)_V, \forall u,v \in V \tag{2.8}$$

With the assumptions (2.2), (2.3) and (2.4a), we have:

$$\forall v \in V, \varphi(v,\cdot) \text{ is convex and l.s.c.} \tag{2.9}$$

$$\varphi(u_1,v_2) - \varphi(u_1,v_1) + \varphi(u_2,v_1) - \varphi(u_2,v_2) \leq \left[(L' + \alpha c_0)\|u_1 - u_2\|_V + M_{\mathcal{R}}\int_0^t \|u_1(s) - u_2(s)\|_V\, ds\right]\|v_1 - v_2\|_V \tag{2.10}$$

Where $M_{\mathcal{R}} = \|\mathcal{R}\|_{\mathcal{C}([0,T];\mathcal{L}(V))}$.

We consider the operator $\mathcal{P}: \mathcal{C}([0,T];V) \to \mathcal{C}([0,T];V)$ such that

$$\mathcal{P}w(t) = \int_0^t w(s)ds + u_0, \quad \forall w \in \mathcal{C}([0,T];V), t \in [0,T]. \tag{2.11}$$

For any function $w \in \mathcal{C}([0,T];V)$, there exists a function $u: [0,T] \to V$ such that

$$u(t) = \mathcal{P}w(t), \qquad \forall t \in [0,T]. \tag{2.12}$$

Proposition 2.1. With (2.8), (2.11) and (2.12). Then, $u \in \mathcal{C}^1([0,T];V)$ is a solution to the problem (1.2a)- (1.2b) if and only if $w \in \mathcal{C}([0,T];V)$ is a solution to the history dependent quasi-variational inequality

$$\big(Bw(t), v - w(t)\big)_V + \varphi(\mathcal{P}w(t),v) - \varphi\big(\mathcal{P}w(t),w(t)\big) \tag{2.13}$$

$$\geq \left(f(t), v - w(t)\right)_V, \forall v \in V.$$

3. Existence and Uniqueness Result

Theorem 3.1. Under the assumptions (2.1a), (2.1b), (2.6), (2.9) and (2.10), the history dependent quasi-variational inequality (2.13) has a unique solution $w \in \mathcal{C}([0,T];V)$.

Proof. The proof of Theorem 3.1 is based on a fixed point argument and will be established in three steps.

3.1 First Step

In the first step let $\eta \in \mathcal{C}([0,T];V)$ be given and denote by $y_\eta \in \mathcal{C}([0,T];V)$ the function

$$y_\eta(t) = \mathcal{P}\eta(t) \;\; \forall t \in [0,T]. \tag{3.1}$$

We consider the following problem: find $w_\eta : [0,T] \to V$ solution of the inequality:

$$\left(Bw_\eta(t), v - w_\eta(t)\right)_V + \varphi\left(y_\eta(t), v\right) - \varphi\left(y_\eta(t), w_\eta(t)\right) \geq \left(f(t), v - w_\eta(t)\right)_V \quad \forall v \in V. \tag{3.2}$$

We have the following existence and uniqueness result.

Lemma 3.2. Under the assumptions (2.1a), (2.1b), (2.6) and (2.9), the problem (3.2) has a unique solution $w_\eta \in \mathcal{C}([0,T];V)$.

Proof. The problem (3.2) is a variational inequality of the second kind. With the assumptions (2.1a), (2.1b), (2.6) and (2.9), the problem (3.2) has a unique solution $w_\eta(t)$ for each $t \in [0,T]$. Let us show that $w_\eta : [0,T] \to V$ is continuous and, to this end, consider $t_1, t_2 \in [0,T]$. For the sake of simplicity in writing we denote $w_\eta(t_i) = w_i$, $\eta(t_i) = \eta_i$, $y(t_i) = y_i$, $f(t_i) = f_i$ for $i = 1,2$.

Using (3.2) we obtain

$$(Bw_1, v - w_1)_V + \varphi(y_1, v) - \varphi(y_1, w_1) \geq (f_1, v - w_1)_V \;\forall v \in V, \tag{3.3}$$

$$(Bw_2, v - w_2)_V + \varphi(y_2, v) - \varphi(y_2, w_2) \geq (f_2, v - w_2)_V \;\forall v \in V. \tag{3.4}$$

We take $v = w_2$ in (3.3) and $v = w_1$ in (3.4), then we add the resulting inequalities to find that

$$(Bw_1 - Bw_2, w_1 - w_2)_V \leq \varphi(y_1, w_2) - \varphi(y_1, w_1) + \varphi(y_2, w_1) - \varphi(y_2, w_2) + (f_1 - f_2, w_1 - w_2)_V.$$

Next, we use assumptions (2.1a) and (2.10) to obtain

$$m\|w_1 - w_2\|_V \leq (L' + \alpha c_0)\|y_1 - y_2\|_V + M_{\mathcal{R}} \int_0^t \|y_1(s) - y_2(s)\|_V \, ds + \|f_1 - f_2\|_V \tag{3.5}$$

We deduce from (3.5) that $t \mapsto w_\eta(t)$ is a continuous function on $[0,T]$.

3.2 Second Step

In the second step we use Lemma 3.2 to consider the operator $\Lambda : \mathcal{C}([0,T];V) \to \mathcal{C}([0,T];V)$ defined by the equality

$$\Lambda\eta = w_\eta, \qquad \forall \eta \in \mathcal{C}([0,T];V). \tag{3.6}$$

We have the following fixed point result.

Lemma 3.3. The operator Λ has a unique fixed point $\eta^* \in \mathcal{C}([0,T];V)$.

Proof. Let $\eta_1, \eta_2 \in \mathcal{C}([0,T];V)$ and let y_i be the function defined by (3.1) for $\eta = \eta_i$, i.e. $y_i = y_{\eta_i}$, for $i = 1,2$. We also denote by w_i the solution of the variational inequality (3.2), for $\eta = \eta_i$, i.e. $w_i = w_{\eta_i}$, $i = 1,2$. Let $t \in [0,T]$. From definition (3.6) we have

$$\|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|_V = \|w_1(t) - w_2(t)\|_V. \tag{3.7}$$

Moreover, an argument similar to that in the proof of (3.5) shows that

$$m\|w_1 - w_2\|_V \leq (L' + \alpha c_0)\|y_1 - y_2\|_V + M_{\mathcal{R}} \int_0^t \|y_1(s) - y_2(s)\|_V \, ds \tag{3.8}$$

We use (3.1) and the definition (2.11) of the operator $\mathcal{P}$ to see that

$$\|y_1(t) - y_2(t)\|_V = \|\mathcal{P}\eta_1(t) - \mathcal{P}\eta_2(t)\|_V \leq \int_0^t \|\eta_1(s) - \eta_2(s)\|_V \, ds$$

and using this inequality in (3.8) yields

$$\|w_1(t) - w_2(t)\|_V \leq \mu \int_0^t \|\eta_1(s) - \eta_2(s)\|_V\, ds. \quad (3.9)$$

where $\mu = (L' + \alpha c_0 + TM_{\mathcal{R}})/m$.

We now combine (3.7) and (3.9) to see that

$$\|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|_V \leq \mu \int_0^t \|\eta_1(s) - \eta_2(s)\|_V\, ds. \quad (3.10)$$

For all $p > 1$, consider the operator $\Lambda^p: \mathcal{C}([0,T];V) \to \mathcal{C}([0,T];V)$, we have:

$$\|\Lambda^p\eta_1(t) - \Lambda^p\eta_2(t)\|_V \leq \frac{\mu^p T^p}{p!}\|\eta_1(t) - \eta_2(t)\|_V.$$

So we have:

$$\|\Lambda^p\eta_1 - \Lambda^p\eta_2\|_{\mathcal{C}([0,T];V)} \leq \frac{\mu^p T^p}{p!}\|\eta_1 - \eta_2\|_{\mathcal{C}([0,T];V)}. \quad (3.11)$$

Since $\frac{\mu^p T^p}{p!} \underset{p\to+\infty}{\to} 0$. Then, the inequality (3.11) implies that a power Λ^p of Λ is a contraction in $\mathcal{C}([0,T];V)$ for p large enough. It follows now from Banachs fixed point theorem that there exists a unique element $\eta^* \in \mathcal{C}([0,T];V)$ such that $\Lambda^p\eta^* = \eta^*$. Moreover, since $\Lambda^p(\Lambda\eta^*) = \Lambda(\Lambda^p\eta^*) = \Lambda\eta^*$, we deduce that $\Lambda\eta^*$ is also a fixed point of the operator Λ^p. By the uniqueness of the fixed point, we conclude that $\Lambda\eta^* = \eta^*$, which shows that η^* is a fixed point of Λ. The uniqueness of the fixed point of the operator Λ follows from the uniqueness of the fixed point of the operator Λ^p.

3.3 The Last Step

In the last step, we have now all the ingredients to prove Theorem 3.1.

Existence: Let $\eta^* \in \mathcal{C}([0,T];V)$ be the fixed point of the operator Λ, i.e. $\Lambda\eta^* = \eta^*$. It follows from (3.1) and (3.6) that, for all $t \in [0,T]$, the following equalities hold:

$$y_{\eta^*}(t) = \mathcal{P}\eta^*(t), \qquad w_{\eta^*}(t) = \eta^*(t). \quad (3.12)$$

We now write the inequality (3.2) for $\eta = \eta^*$ and then use the equalities (3.12) to conclude that the function $\eta^* \in \mathcal{C}([0,T];V)$ is a solution to the quasi-variational inequality (2.13).

Uniqueness: Denote by $\eta^* \in \mathcal{C}([0,T];V)$ the solution of the quasi-variational inequality (2.13), and let $\eta \in \mathcal{C}([0,T];V)$ be a different solution of this inequality. Also, consider the function $y_\eta \in \mathcal{C}([0,T];V)$ defined by (3.1). Then it follows from (2.13) that η is solution to the variational inequality (3.2) and, since by Lemma 3.2 this inequality has a unique solution, denoted w_η, we conclude that

$$\eta = w_\eta. \quad (3.13)$$

Equality (3.13) shows that $\Lambda\eta = \eta$, where Λ is the operator defined by (3.6). Therefore, by Lemma 3.3 it follows that $\eta = \eta^*$.

4. A Convergence Result

In this section, we proceed with a result concerning the dependence of the solution of the history slip-dependent evolutionary quasi-variational inequality (1.2a)-(1.2b) with respect to the operator A, the relaxation operator $\mathcal{R}$ and the function j. To this end, we assume in what follows that (1.2a)-(2.7) hold. We denote by u the solution of problem (1.2a)-(1.2b). Also, for each $\rho > 0$ we consider a perturbation $A_\rho: V \to V$ of A which satisfies condition (2.2) with $L'_\rho > 0$, a perturbation $\mathcal{R}_\rho: [0,T] \to \mathcal{L}(V)$ of $\mathcal{R}$ and a perturbation $j_\rho: Y \times V \to \mathbb{R}$ of j which satisfies (2.4b) with $\alpha_\rho > 0$.

We consider the problem of finding a function $u_\rho: [0,T] \to V$ such that

$$\left(Au_\rho(t), v - \dot{u}_\rho(t)\right)_V + \left(B_\rho\dot{u}_\rho(t), v - \dot{u}_\rho(t)\right)_V + \left(\int_0^t \mathcal{R}_\rho(t-s)u_\rho(s)ds, v - \dot{u}_\rho(t)\right)_V + j(\mathcal{S}(u_\rho), v) - j\left(\mathcal{S}(u_\rho), \dot{u}_\rho(t)\right) \quad (4.1)$$

$$\geq \left(f(t), v - \dot{u}_\rho(t)\right)_V \quad \forall v \in V, t \in [0,T],$$

$$u_\rho(0) = u_0. \tag{4.2}$$

We deduce from Proposition 2.1 and Theorem 3.1 that problem (4.1)-(4.2) has a unique solution $u_\rho \in \mathcal{C}^1([0,T];V)$.

Consider now the following assumptions.

$$\left.\begin{array}{l}\text{There exists } F:\mathbb{R}_+ \to \mathbb{R}_+ \text{ such that}\\ (a)\left\|A_\rho(v) - A(v)\right\|_V \leq F(\rho)\\ \quad \forall v \in V, \forall \rho > 0,\\ (b)\ \lim_{\rho\to 0} F(\rho) = 0.\end{array}\right\} \tag{4.3}$$

$$\left.\begin{array}{l}\text{There exists } G:\mathbb{R}_+ \to \mathbb{R}_+ \text{ such that}\\ (a)\left\|\mathcal{R}_\rho v - \mathcal{R}v\right\|_{\mathcal{C}([0,T];\mathcal{L}(V))} \leq G(\rho)\\ \quad \forall v \in V, \forall \rho > 0,\\ (b)\ \lim_{\rho\to 0} G(\rho) = 0.\end{array}\right\} \tag{4.4}$$

$$\left.\begin{array}{l}\text{There exists } H:\mathbb{R}_+ \to \mathbb{R}_+ \text{ and } c > 0:\\ (a)\ j_\rho(y_1, v_2) - j_\rho(y_1, v_1)\\ \quad + j(y_2, v_1) - j(y_2, v_2)\\ \quad \leq H(\rho)\|v_1 - v_2\|_V\\ + c\|y_1 - y_2\|_Y \|v_1 - v_2\|_V\\ \quad \forall y_1, y_2 \in Y, v_1, v_2 \in V, \forall \rho > 0,\\ (b)\ \lim_{\rho\to 0} H(\rho) = 0.\end{array}\right\} \tag{4.5}$$

The behavior of the solution u_ρ as $\rho \to 0$ is given by the following theorem.

Theorem 4.1. Under the assumptions (4.3)-(4.5) the solution u_ρ of problem (4.1)-(4.2) converges to the solution u of problem (1.2a)-(1.2b), i.e.,

$$u_\rho \to u \text{ in } \mathcal{C}^1([0,T];V) \text{ as } \rho \to 0.$$

Proof. Let $\rho > 0$ and denote by w_ρ and w the functions defined by

$$w_\rho(t) = \dot{u}_\rho(t), w(t) = \dot{u}(t) \quad \forall t \in [0,T]. \tag{4.6}$$

and let a perturbation functional $\varphi_\rho: V \times V \to \mathbb{R}$ of φ given by

$$\varphi_\rho(u,v) = \left(A_\rho u, v\right)_V + j_\rho(\mathcal{S}(u), v) + \left(\int_0^t \mathcal{R}_\rho(t-s)u(s)ds, v\right)_V, \quad \forall u, v \in V. \tag{4.7}$$

Using Proposition 2.1, the problem (4.1)-(4.2) is equivalent to the following inequality

$$\left(Bw_\rho(t), v - w_\rho(t)\right)_V + \varphi_\rho\left(\mathcal{P}w_\rho(t), v\right) - \varphi_\rho\left(\mathcal{P}w_\rho(t), w_\rho(t)\right) \geq \left(f(t), v - w_\rho(t)\right)_V. \tag{4.8}$$

We take $v = w(t)$ in (4.8) and $v = w_\rho(t)$ in (2.13). Then we add the resulting inequalities to obtain

$$\begin{aligned}\left(Bw_\rho(t) - Bw(t), w_\rho(t) - w(t)\right)_V &\leq \varphi_\rho\left(\mathcal{P}w_\rho(t), w(t)\right)\\ &- \varphi_\rho\left(\mathcal{P}w_\rho(t), w_\rho(t)\right)\\ &+ \varphi\left(\mathcal{P}w(t), w_\rho(t)\right)\\ &- \varphi\left(\mathcal{P}w(t), w(t)\right)\end{aligned}$$

Next, we use the assumptions (2.1a), (4.3)(a), (4.4)(a), (4.5)(a),(4.7), (2.5) and (2.11) to see that

$$\begin{aligned}m\left\|w_\rho(t) - w(t)\right\|_V &\leq F(\rho) + TG(\rho) + H(p)\\ &+ (L' + cc_0\\ &+ TM_{\mathcal{R}})\int_0^t \left\|w_\rho(s) - w(s)\right\|_V ds.\end{aligned}$$

With the Gronwall inequality [6, p. 60], we have:

$$\left\|w_\rho(t) - w(t)\right\|_V \leq \frac{c}{m}\left(F(\rho) + TG(\rho) + H(\rho)\right).$$

Using (4.3)(b), (4.4)(b) and (4.5)(b), we obtain

$$w_\rho \to w \quad \text{in } \mathcal{C}([0,T];V) \text{ as } \rho \to 0 \tag{4.9}$$

From (4.6) and (4.2), we obtain

$$u_\rho(t) - u(t) = \int_0^t \left(w_\rho(s) - w(s)\right)ds, \quad \forall t \in [0,T]. \tag{4.10}$$

Using (4.6), (4.9) and (4.10) we find

$$\dot{u}_\rho \to \dot{u} \text{ in } \mathcal{C}([0,T];V) \text{ as } \rho \to 0,$$
$$u_\rho \to u \text{ in } \mathcal{C}([0,T];V) \text{ as } \rho \to 0.$$

which concludes the proof.

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References

[1] A. Rodriguez-Aros, M. Sofonea, J.M. Viano, A class of evolutionary variational inequalities with Volterra-type integral term, Mathematical Models and Methods in Applied Sciences (M3AS) 14 (2004) 555-577.

[2] A. Rodriguez-Aros, M. Sofonea, J.M. Viano, Numerical analysis of a class of evolutionary variational inequalities with integral term, Ann. Univ. Craiova, Math. Comp. Sci. 30 (2003) 30-37.

[3] M. Sofonea, A. Rodriguez-Aros, J. M. Viano, A class of integro-differential variational inequalities with applications to viscoelastic contact, Mathematical and Computer Modelling 41 (2005) 1355-1369.

[4] M. Raous, M. Jean, J.J. Moreau, Contact Mechanics, Plenum Press, New York, 1995.

[5] A. Rodriguez-Aros, M. Sofonea, J.M. Viano. Numerical analysis of a frictional contact problem for viscoelastic materials with long-term memory, Numerische Mathematik 198 (2007), 327-358.

[6] M. Sofonea, A. Matei, Mathematical Models in Contact Mechanics, Cambridge University Press, 2012.

[7] M. Sofonea, A. Matei. Variational Inequalities with Applications: A Study of Antiplane Frictional Contact Problems. Springer, New York, 2009.

[8] M. Sofonea, A. Matei. History-dependent quasivariational inequalities arising in Contact Mechanics, European J. Appl. Math. 22 (2011) 471-491.