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A New Approach to One Parameter Motion

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Abstract: In our study, we study a different approach to one parameter motion. We think that while one of the planes is fixed, the other is deformation on the plane with shear motion. By this way, we will calculate the velocity connection and pole curve that occurred by the movement. Planar motion, pole curve, shear mapping.

Keywords: Planar motion, pole curve, shear mapping.

1. Introduction

Let us consider one parameter plane motion of the moving plane E with respect to the fixed plane E'. $\{O,\vec{e}_1,\vec{e}_2\}$ and $\{O',\vec{e}_1{}',\vec{e}_2{}'\}$ be their coordinate system, respectively. The motion defined by the rotation transformation of the base vectors

$$\vec{e}_1=\vec{e}_1{}' \cos\varphi+\vec{e}_2{}' \sin\varphi$$
$$\vec{e}_2=-\vec{e}_1{}' \sin\varphi+\vec{e}_2{}' \cos\varphi$$

and the translation transformation between planes

$$\vec{u}=\vec{e}_1\, u_1+\vec{e}_2\, u_2$$

is called one-parameter plane motion and denoted by $H=E/E'$, where x and x' are the position vectors with respect to the moving and fixed coordinate systems of a point $X\in E$.

The derivation equations of the one parameter motion is

$$\dot{\vec{e}}_1=(-\vec{e}_1{}' \sin\varphi+\vec{e}_2{}' \cos\varphi)\dot{\varphi}$$

$$\dot{\vec{e}}_2=-(\vec{e}_1{}' \cos\varphi+\vec{e}_2{}' \sin\varphi)\dot{\varphi}$$

where "." denotes the derivation with respect to t. The differentiation of the position vector x with respect to t yields the relative velocity vector

$$\vec{v_r}=\dot{\vec{x}}=\vec{e}_1\,\dot{x}_1+\vec{e}_2\,\dot{x}_2$$

and the differentiation of the position vector x' with respect to t yields the absolute velocity vector

$$\vec{v_a}=\dot{\vec{x'}}=\vec{e}_1\,(-\dot{\vec{u}}_1+u_2\dot{\varphi}-x_2\dot{\varphi})+\vec{e}_2\,(-\dot{u}_2-u_1\dot{\varphi}+x_1\dot{\varphi})+\vec{e}_1\,\dot{x}_1+\vec{e}_2\,\dot{x}_2$$

Also we get the dragging vector as following:

$$\vec{v_f}=\dot{\vec{x'}}=-\vec{e}_1\,[\dot{\vec{u}}_1+(-u_2+x_2)\dot{\varphi}]+\vec{e}_2\,[-\dot{u}_2+(-u_1+x_1)\dot{\varphi}]$$

Thus, the absolute velocity vector is equal to the sum of relative velocity and dragging velocity, i.e.:

$$\vec{v_a}=\vec{v_f}+\vec{v_r}$$

The instantaneous pole of one parameter motion is

$$p_1=u_1+\frac{\dot{u}_2}{\dot{\varphi}}p_2=u_2-\frac{\dot{u}_1}{\dot{\varphi}}$$

in the references [1-2]. Also the instantaneous pole for a motion and other theoretical properties of motion was given in [3-8].

In kinematic, due to the coordinate system the one parameter planar motion is defined by the rotation and translation transformation. Also the planes which are used in planar motion are the same each other. Theoretically, this is possible but under some physical circumstances, there may be a deformation in the structure of the plane. If we take plane as a metal plate, it can be deformed from the heat or if the plane is made of plastic it can be deformed from the stretching. If we apply shear translation on (x,y) point, the

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transformed point will be $(x', y') = (x, ax + y)$ in [9-12]. In here, the notation a gives us the shearing angle. Also, the base vectors of the plane are transformed under shear in our study. Later, we will define translation vectors between two planes. Thus motion is made up from shearing and translation motion.

In this study, we study a different approach to one parameter motion. The movement of the fixed point that is taken on the moving plane is defined by the both rotation and translation transformation. Nevertheless, while analyzing the motion of planes relative to each other, the deformation isn't considered. Here, we explain the shearing deformations that are occurred with geometry. Now here, we are defining the motions with shearing and translation. In terms of making some differences on planar motion method, we are able to look varied on kinematic.

2. One Parametric Motion by Shearing Transformation

In our motion, E moving plane which is not orthogonal be the $\{O, \vec{e}_1, \vec{e}_2\}$ as unit frame. If E' is the orthonormal frame of the fixed plane then it is $\{O', \vec{e}_1{}', \vec{e}_2{}'\}$. We will accept these two coordinate systems as the representative of E and E' planes and we will talk about the one parametric motion of the moving coordinate system due to the fixed coordinate system. In our study, we will define planar motion with the translation and shearing motions. The translation vector which goes from the beginning point of the moving system to the beginning point of the fixed system is as:

$$\overrightarrow{OO}' = \vec{u} = \vec{e}_1 u_1 + \vec{e}_2 u_2.$$

The angle which is done to each other coordinate systems will be taken as the shearing coefficient. In that case, we can define the unit base vectors of the moving plane with shearing transformation as the following:

$$\vec{e}_1 = \vec{e}_1{}' \tag{2.1}$$

$$\vec{e}_2 = \vec{e}_1{}' \frac{\varphi}{\sqrt{1+\varphi^2}} + \vec{e}_2{}' \frac{1}{\sqrt{1+\varphi^2}} \tag{2.2}$$

Definition 2.1. Defined by the u translation and the φ shear transformation E and E' planar motion is called "***one parameter planar motion with shear transformation***" and it is shown like that $H_{S1} = E/E'$.

Let u_1, u_2, φ magnitudes be the t real parameter's that arising sufficient degree, and here t parameter is accepted as time. So these functions

$$u_1 = u_1(t), \quad u_2 = u_2(t), \quad \varphi = \varphi(t)$$

are declared. If (x_1, x_2) and $(x_1{}', x_2{}')$ are the coordinates of X point which are taken on the moving E plane and fixed E' plane then the location vectors that corresponding to these points can be defined as

$$\vec{X} = \overrightarrow{OX} = \vec{e}_1 x_1 + \vec{e}_2 x_2$$
$$\vec{X}' = \overrightarrow{O'X} = \vec{e}'_1 x'_1 + \vec{e}'_2 x'_2.$$

In this case, from the vectorial addition rule following correlations can be obtained:

$$\overrightarrow{O'X} = \overrightarrow{O'O} + \overrightarrow{OX} = -\overrightarrow{OO'} + \overrightarrow{OX} \tag{2.3}$$

$$\vec{x} = -\vec{u} + \vec{x} = \vec{e}_1(-u_1 + x_1) + \vec{e}_2(-u_2 + x_2). \tag{2.4}$$

If the x_1, x_2 coordinates of X point are independent from t timing so x_1, x_2 are fixed then we will think about X point is determined on E plane or shortly we call X is a point of E. Adversely if the $x_1{}', x_2{}'$ are the fixed points then X stays fix on E or X will be a point of E.

Example: In this numeric example, the moving plane E moves about the starting point

$$(tcost + 2tsint, tsint - 2tcost)$$

with the shearing angle $\varphi(t) = t$ on the fixed plane E'. Let's make some comparisons of result pole curves by finding the planar motion with shear transformations $(Fig - a)$ and also the usual planar motion $(Fig - b)$ with using the program MAPLE11. The translation vector is shown in Fig.1.

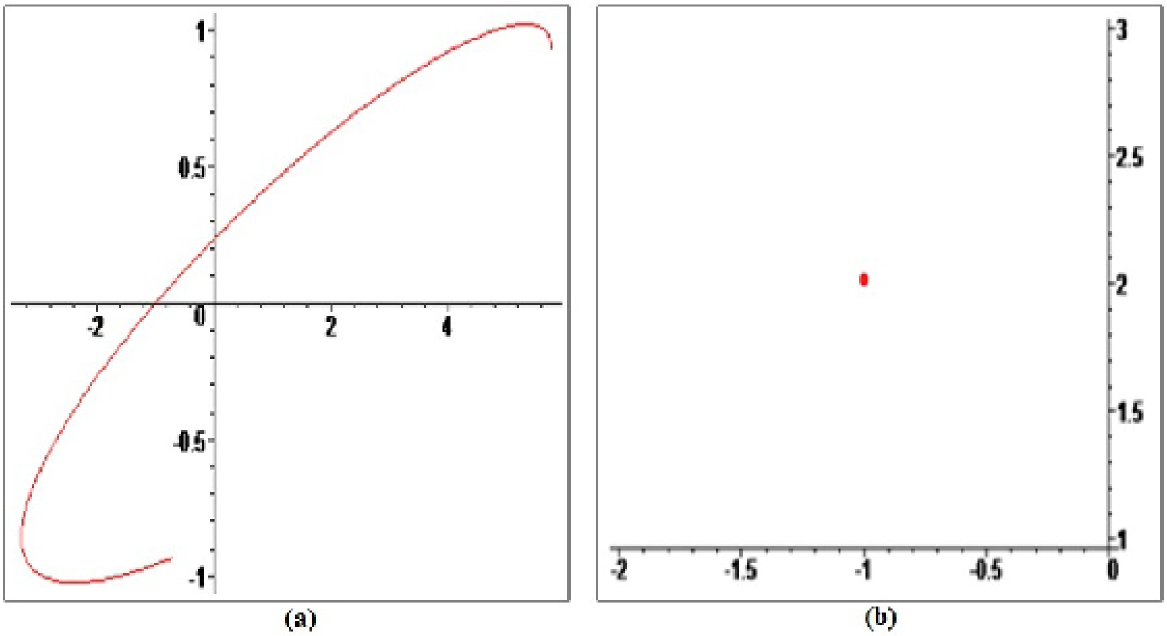


Fig. 1 **The output of the translation vector $\vec{u}$.**

3. Derivative Equations

For searching the X point's speed due to the both fixed and moving plane, we should gain the derivative equations of motion. For this reason, let's accept the $\vec{e}_1{}'$ and $\vec{e}_2{}'$ vectors are fix and calculate the derivations of Equation (2.1) and (2.2) according to t:

Theorem 3.1. Defined with the shearing transformation $H_{S1} = E/E'$ planar motion of the base vectors are the derivative which is shown

$$\dot{\vec{e}}_1 = 0 \tag{3.5}$$

$$\dot{\vec{e}}_2 = \frac{\dot{\varphi}}{\sqrt{1+\varphi^2}}\vec{e}_1 - \frac{\varphi\dot{\varphi}}{1+\varphi^2}\vec{e}_2. \tag{3.6}$$

Proof: While we are differentiating due to t on E plane we should think $\vec{e}_1{}'$ and $\vec{e}_2{}'$ base vectors are fixed. Thus, derivative equations can be find as the following:

$$\dot{\vec{e}}_1 = 0$$

$$\dot{\vec{e}}_2 = \dot{\varphi}\left(\frac{\vec{e}_1}{\sqrt{1+\varphi^2}} - \frac{\varphi\,\vec{e}_2}{1+\varphi^2}\right).$$

Definition 3.1. The derivative $\dot{\varphi}$ which passes in moving derivative equation is called '***angular velocity***' of the shearing transformation.

Theorem 3.2. $H_{S1} = E/E'$ with one parameter planar motion due to t time, derivative will be given with following equation:

$$\dot{\vec{u}} = \vec{e}_1\left(\dot{u}_1 + \frac{\dot{\varphi}u_2}{\sqrt{1+\varphi^2}}\right) + \vec{e}_2\left(\frac{\varphi\dot{\varphi}}{1+\varphi^2}u_2 + \dot{\vec{u}}_2\right). \tag{3.7}$$

Proof: While differentiating of the vector $\vec{u} = \vec{e}_1 u_1 + \vec{e}_2 u_2$, it is clear that the derivative equations will be calculated easily. Here the bases vectors $\vec{e}_1$ and $\vec{e}_2$ are on moving plane, and also are not fixed. While performing one parametered-motion on E plane according to E' plane, a X point can change its position on moving E plane with the t time. Thus, composing from the motion three different kind of velocity occurs they are relative, absolute and dragging velocity vectors, in Fig.2.

Definition 3.2. The velocity vector of X point which belongs to (X) curve on E is called as "***relative velocity vector of X point***" and is shown with $\vec{v}_r$. So, we can give relative velocity vector (in Fig.3) with the equation

$$\vec{v}_r=\dot{\vec{X}}=\vec{e}_1\dot{\vec{x}}_1+\vec{e}_2\dot{\vec{x}}_2.$$

Definition 3.3. The velocity vector of X' point which belongs to (X') curve on E is called as "***absolute velocity vector of X′ point***" and is shown with $\vec{v}_a$. So, we can give absolute velocity vector with the equation

$$\vec{v}_a=-\dot{\vec{u}}+\dot{\vec{x}}.$$

Definition 3.4. While X point is fixed on $H_{S1=E/E'}$ planar motion, that means on E(i.e. $\vec{v}_r=\vec{0}$) moving planar, the velocity vector of X point is called as "***the dragging velocity of X point***" and is shown with $\vec{v}_f$, see in Fig.4.

Theorem 3.3. The absolute velocity vector in $H_{S1}=E/E'$ motion is equal to the sum of relative velocity and dragging velocity. So, the union rule of velocity vectors can be shown with the equation in the following:

$$\vec{v}_a=\vec{v}_f+\vec{v}_r$$

Proof: With the use of the Equations (3.5), (3.6) and (3.7), the derivative of the Equations (2.4) and (2.3) according to the t parameter is as the following:

$$\vec{v}_a=-\dot{u}+\dot{x}$$
$$=-\vec{e}_1\left(\dot{u}_1+\frac{\dot{\varphi}u_2}{\sqrt{1+\dot{\varphi}^2}}\right)-\vec{e}_2\left(\frac{\dot{\varphi}\varphi}{1+\varphi^2}u_2+\dot{\vec{u}}_2\right)$$
$$+\dot{\varphi}\left(\frac{\vec{e}_1}{\sqrt{1+\varphi^2}}\right)x_2$$

Due to the Definition (3.4), when $\vec{v}_r=\vec{0}$, the dragging velocity vector is in the form of

$$\vec{v}_f=\vec{e}_1\left[-\dot{u}_1-\frac{\dot{\varphi}u_2}{\sqrt{1+\dot{\varphi}^2}}+\frac{\dot{\varphi}x_2}{\sqrt{1+\varphi^2}}\right]$$
$$+\vec{e}_2\left[\frac{-\dot{\varphi}\varphi}{1+\varphi^2}u_2-\dot{\vec{u}}_2-\frac{\dot{\varphi}\varphi x_2}{1+\varphi^2}\right],$$

and thus we get the relation of the velocities

$$\vec{v}_a=\vec{v}_f+\vec{v}_r.$$

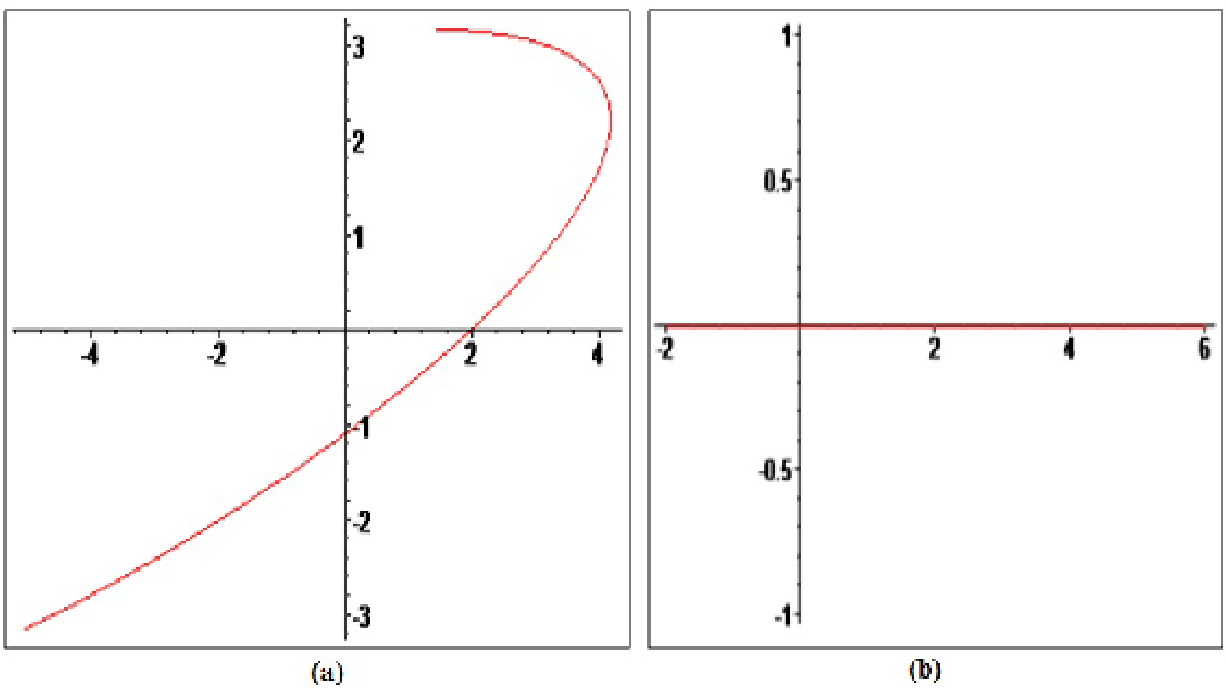
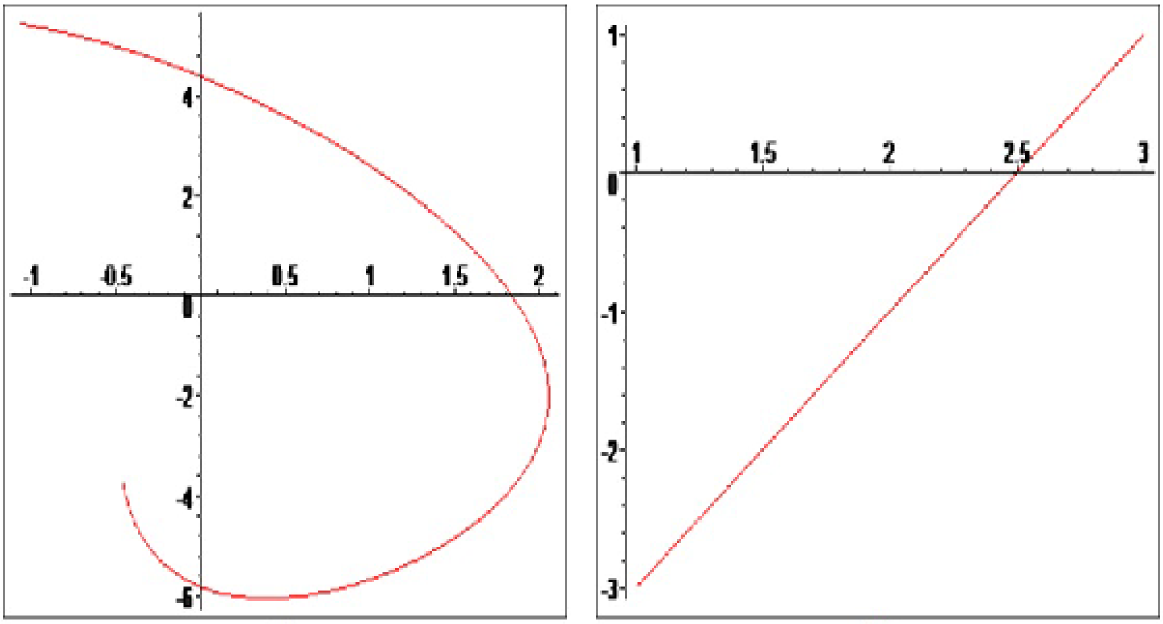
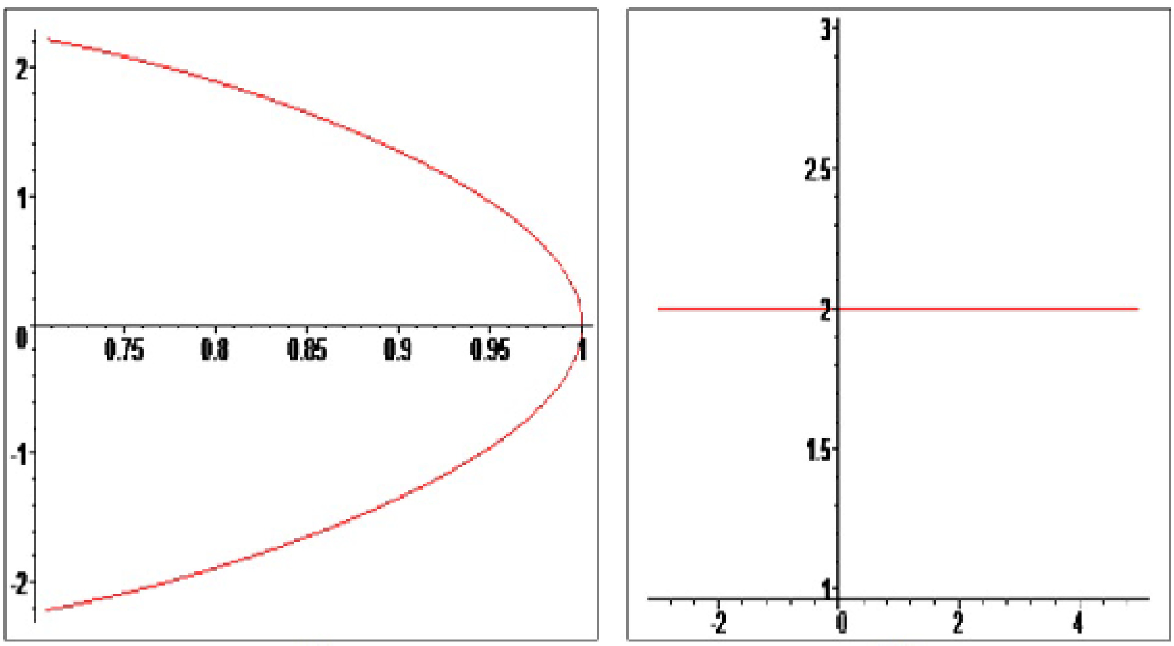


Fig. 2 The output of the relative velocity vector $\vec{v}_r$.



(a) (b)

Fig. 3 The output of the absolute velocity vector $\vec{v}_a$



(a) (b)

Fig. 4 The output of the dragging velocity vector $\vec{v}_f$.

4. Instantaneous Shearing Pole and Pole Curves

Here, we will research the points, which have zero dragging velocity in each t moment, of our motion with one parameter.

Theorem 4.1. For each immediate t parameter, the dragging velocity vector is zero in only one point, it means that there is only one point in each E and E' planes with $\dot{\varphi} \neq 0$ limitation on $H_{S1} = E/E'$ motion.

Proof: In $H_{S1} = E/E'$ motion, the point, on which the dragging velocity is zero for each t parameter, is not only stable in E plane but also stable in E' plane. The dragging vector is equal to zero as $\vec{v}_f = \vec{0}$. So, we obtain the equations in the following:

$$-\dot{u}_1 - \frac{\dot{\varphi} u_2}{\sqrt{1+\dot{\varphi}^2}} + \frac{\dot{\varphi} x_2}{\sqrt{1+\varphi^2}} = 0 \qquad (4.8)$$

$$\frac{-\dot{\varphi}\varphi}{1+\varphi^2} u_2 - \dot{\vec{u}}_2 - \frac{\dot{\varphi}\varphi x_2}{1+\varphi^2} = 0 \qquad (4.9)$$

Here, due to $\dot{\varphi} \neq 0$ and $p \neq 0$, we find the relations in the following from the Equations(4.8) and (4.9). Since $\dot{\vec{e}} = 0$ in the shearing movement, it is understood that $\vec{e}_1$ base is fixed during the motion. So, x_1 abscissa on two planes is in the immediate motions of fixed and moving motions. Thus here

$$p_1 = x_1$$

$$p_2 = \frac{\sqrt{1+\dot{\varphi}^2}}{\dot{\varphi}} \dot{u}_1 + u_2 = -\frac{1+\varphi^2}{\dot{\varphi}\varphi} \dot{u}_2 - u_2$$

equations give us pole points.

Definition 4.1. $P(p_1, p_2)$ point given with $\overrightarrow{OP} = \vec{P} = \vec{e}_1\, p_1 + \vec{e}_2\, p_2$ position vector is called as the pole of H_{S1} motion or the instantaneous shearing pole or pole center.

Remark 4.1. Also we can state the dragging velocity with the components of pole point if we write $\dot{u}_1$ and $\dot{u}_2$ in the dragging velocity. So,

$$\vec{v}_f = (x_2 - p_2)\left[\frac{1}{\sqrt{1+\dot{\varphi}^2}} \vec{e}_1 - \frac{\varphi}{1+\varphi^2}\right]\dot{\varphi}$$

we can obtain this equation.

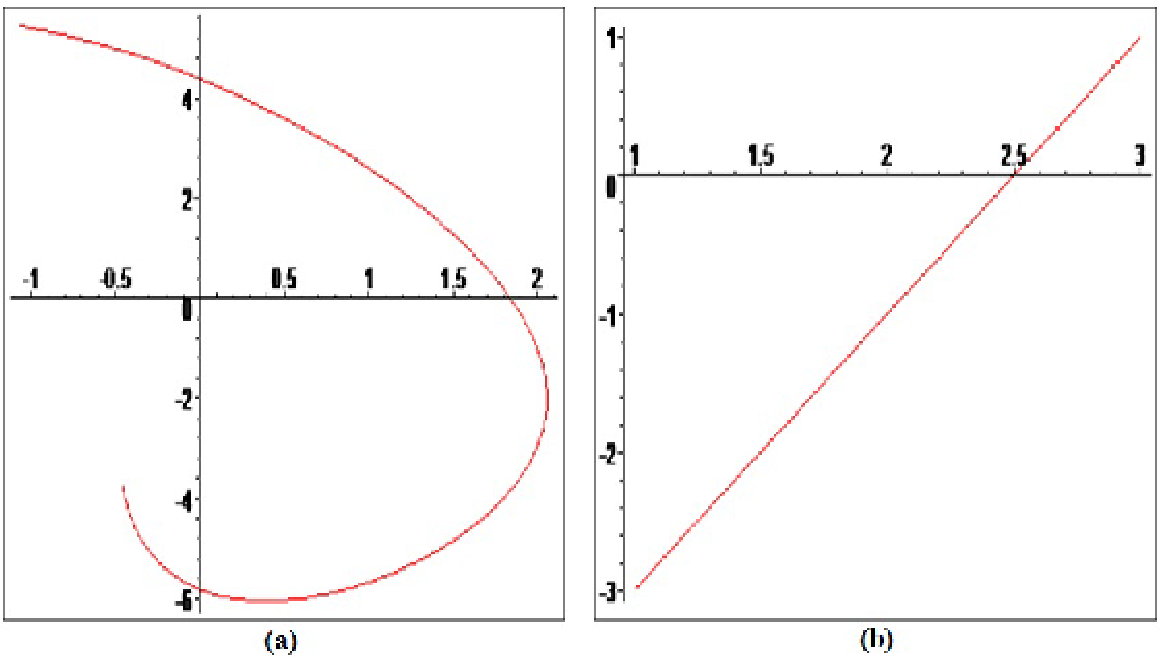


Fig. 5 The output of the Pole curves.

5. Results

(1) The X point that is taken on the moving coordinate plane at time t and the vector which is composed by the P pole point at the same time is

$$\overrightarrow{PX} = \vec{e}_2\,(x_2 - p_2).$$

$\overrightarrow{PX}$ vector's $\vec{v}_f$ dragging vector for not being zero, it is seen these vectors won't be perpendicular. So, formed by the help of shear transformation

$$<\overrightarrow{PX}, \vec{v}_f> = -\frac{(x_2 - p_2)^2 \varphi}{1+\varphi^2}$$

equation is provided at planar motion.

(2) The length of $\vec{v}_f$ dragging velocity vector, planar motion of shear transformation is as the following:

$$|\ \vec{v}_f\ | = |x_2 - p_2|\dot{\varphi}\sqrt{\frac{1}{\sqrt{1+\dot{\varphi}^2}} + \frac{\varphi^2}{(1+\varphi^2)^2}}$$

(3) On one parameter planar motion H_{S1} with shearing transformation, the moving pole curve (P) of the plane E doesn't roll onto fixed pole curve $(P^{'})$ of the plane $E^{'}$ without sliding.

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