Journal of Mathematical Sciences 2 (2015) 211-216
ISSN 2372-5214
Betty Jones & Sisters Publishing
http://www.bettyjonespub.com

The Skew Inverse Semigroup Ring

S. Moayeri Rahni

Department of Mathematics, College of Sciences, Shiraz University, Shiraz, 71454, Iran

Abstract: Given a partial action π of an inverse semigroup S on an associative algebra $\mathcal{A}$, we introduce the skew inverse semigroup ring $\mathcal{A} \rtimes_{\pi}$ S. We will show that under some condition this construction is associative. Expressing the notion of enveloping action, we will prove that if $(\beta, \mathcal{B})$ is an enveloping action for a partial action $(\pi, \mathcal{A})$ then $\mathcal{A} \rtimes_{\pi}$ S and $\mathcal{B} \rtimes_{\beta}$ S are Morita equivalent.

Keywords: Inverse semigroup, partial action, partial representation, enveloping action.

1. Introduction

The notion of a *partial crossed product* of a C^*-algebra by the group of integers was defined by R. Exel in [1]. Roughly, the automorphism used in the definition of a crossed product of a C^*-algebra $\mathcal{A}$ by the group of integers was replaced by an isomorphism between two ideals of $\mathcal{A}$, namely, partial automorphisms. K. McCalanahan [2] defined a partial crossed product of a C^*-algebra by a discrete group. The ideas involved were mostly straightforward generalization of the definitions given in [1]. K. McCalanahan used partial action of groups to define a partial crossed products. This motivates us to define the *skew inverse semigroup ring* by a partial actions of an inverse semigroups S, and we will discuss the associativity of this construction.

Throughout this work $\mathcal{A}$ will stand for an associative $\mathcal{K}$-algebra, where $\mathcal{K}$ is a unital commutative ring. We recall from [3] that a partial action of an inverse semigroup S on a set X is a partial homomorphism $\pi: S \mapsto \mathrm{I}(X)$, where $\mathrm{I}(X)$ is the semigroup of all partially defined bijection maps on subsets of X, that is, for each $s, t \in S$

$$\pi(s^*)\pi(s)\pi(t) = \pi(s^*)\pi(st), \quad \pi(s)\pi(t)\pi(t^*) = \pi(st)\pi(t^*),$$

but we use [3, Proposition] to give a definition of a partial action of an inverse semigroup.

Definition 1.1. Suppose that S is an inverse semigroup and X is a set. A partial action of S on X is a map $\pi: S \mapsto \mathrm{I}(X)$ satisfying the following conditions:

(i) $\pi_s^{-1} = \pi_{s^*}$,

(ii) $\pi_s(X_{s^*} \cap X_t) = X_s \cap X_{st}$ for all $s, t \in S$ (where X_s denotes the rang of π_s for each $s \in S$),

(iii) $\pi_s(\pi_t(x)) = \pi_{st}(x)$ for all $x \in X_{t^*} \cap X_{t^*s^*}$.

For a partial action π of an inverse semigroup S on an associative $\mathcal{K}$-algebra $\mathcal{A}$, we assume in Definition 1.1 that each X_s $(s \in S)$ is an ideal of $\mathcal{A}$ and that every map $\pi_s: X_{s^*} \mapsto X_s$ is an isomorphism of algebras. Furthermore, if the inverse semigroup S is unital, we shall suppose that $X_1 = \mathcal{A}$ and π_1 is the identity map of $\mathcal{A}$, where 1 is nothing but the unit of S.

Now, we are ready to define the *skew inverse semigroup ring*. Let π be a partial action of S on $\mathcal{A}$. Let $L = \{\sum_{s \in S} a_s \delta_s : a_s \in X_s\}$, the set of all formal finite sums, with the following multiplication:

$$(a_s\delta_s).(b_t\delta_t) = \pi_s(\pi_{s^*}(a_s)b_t)\delta_{st}.$$

Note that $\pi_s(\pi_{s^*}(a_s)b_t)$ is an element of X_{st} simply because

$$\pi_s(\pi_{s^*}(a_s)b_t \in \pi_s(X_{s^*} \cap X_t) = X_s \cap X_{st}.$$

Thus the multiplication is well-defined, and L is an algebra with this multiplication. Let I be the ideal of L generated by the set $\{a\delta_r - a\delta_t :$ where $r \leq$

Corresponding author: S. Moayeri Rahni, Ph.D., research fields: C*-algebras and crossed product. E-mail: smoayeri@shirazu.ac.ir.

t and $a \in X_r\}$. Note that by [3, Proposition 3.8] $X_r \subseteq X_t$ since $r \leq t$. Now, we are at a position to introduce the *skew inverse semigroup ring* $\mathcal{A} \rtimes_\pi S$ associated to the partial action π as we proposed in the beginning of the section. the *skew inverse semigroup ring* $\mathcal{A} \rtimes_\pi S$ is the quotient algebra $\frac{L}{I}$. Considering the quotient map $\Phi: L \to \mathcal{A} \rtimes_\pi S$ from L onto $\mathcal{A} \rtimes_\pi S$, we will denote the element $\Phi(a_s\delta_s) \in \mathcal{A} \rtimes_\pi S$ by $\overline{a_s\delta_s}$. The associativity of this construction was proved in [1] in the context of C^*-algebras, when S is a group. The question that naturally arises is when $\mathcal{A} \rtimes_\pi S$ is associative. This question will be answered in the next section. We will use the multiplier algebra to show that $\mathcal{A} \rtimes_\pi S$ is associative under some condition. One of our goals is to show that $\mathcal{A} \rtimes_\pi S$ is associative whenever $\mathcal{A}$ is semiprime (Corollary 2.4), but $\mathcal{A} \rtimes_\pi S$ is not always associative, for example, see [1, example, 3.5].

In the section 3, we will prove that if $(\beta, \mathcal{B})$ is an enveloping action for partial action $(\pi, \mathcal{A})$ such that $\mathcal{A}$ and $\mathcal{B}$ are unital algebras then $\mathcal{A} \rtimes_\pi S$ and $\mathcal{B} \rtimes_\beta S$ are Morita equivalent. This section contains preliminary results that are needed in the proof of the main theorem of the next section, Theorem 2.1.

Let $\mathcal{K}$ be a unital commutative ring, $\mathcal{A}$ a unital associative $\mathcal{K}$-algebra, and $\mathcal{I}$ an ideal of $\mathcal{A}$. The algebra of *multipliers* on $\mathcal{I}$ is the set $M(\mathcal{I})$ of all ordered pairs (L, R) where L and R are linear transformations on $\mathcal{I}$ such that for all $a, b \in \mathcal{I}$ the followings are valid:

(i) $L(ab) = L(a)b$,

(ii) $R(ab) = aR(b)$,

(iii) $R(a)b = aL(b)$.

For $(L, R), (L', R') \in M(\mathcal{I})$ and $\alpha \in \mathcal{K}$ the algebra's operations are given by:

(i) $\alpha(L, R) = (\alpha L, \alpha R)$,

(ii) $(L, R) + (L', R') = (L + L', R + R')$,

(iii) $(L, R)(L', R') = (L \circ L', R' \circ R)$.

It is easy to see that $M(\mathcal{I})$ is an associative unital algebra with the unit element (L_1, R_1) where L_1, R_1 are identity maps on $\mathcal{I}$. Consider the map $\varphi: \mathcal{I} \mapsto M(\mathcal{I})$ such that $\varphi(x) = (L_x, R_x)$, where $L_x(y) = xy$ and $R_x(y) = yx$ for $y \in \mathcal{I}$. Obviously, φ is an algebra homomorphism.

Definition 1.2. We will say that an algebra $\mathcal{I}$ is *non-degenerate* if the map $\varphi: \mathcal{I} \ni x \mapsto (L_x, R_x)$ is injective.

Note that the kernel of φ is the intersection of all right annihilators of $\mathcal{I}$ in $\mathcal{I}$ with the left annihilators of $\mathcal{I}$ in $\mathcal{I}$, consequently, $\mathcal{I}$ is non-degenerate if for each non-zero element x in $\mathcal{I}$ there exists an element y in $\mathcal{I}$ such that $xy \neq 0$ or $yx \neq 0$. An algebra $\mathcal{I}$ is called *(L,R)-associative* if, for any two multipliers (L, R) and (L', R') in $M(\mathcal{I})$, one has that $R' \circ L = L \circ R'$. For more information about multiplier algebras, we will refer the reader to ([4]-[5]).

2. Associativity of Skew Inverse Semigroup Rings

With the aid of the following pivotal Theorem, we are able to answer the associativity question.

Theorem 2.1. If π is a partial action of a unital inverse semigroup S on an algebra $\mathcal{A}$ such that each X_s $(s \in S)$ is (L, R)-associative, then the skew inverse semigroup ring $\mathcal{A} \rtimes_\pi S$ is associative.

Proof. Obviously, $\mathcal{A} \rtimes_\pi S$ is associative if L is associative, and L is associative if and only if

$$(a\delta_s b\delta_t)c\delta_r = a\delta_s(b\delta_t c\delta_r) \tag{1}$$

for any $s, t, r \in S$, $a \in X_s$, $b \in X_t$, and $c \in X_r$. By considering the left hand side of the (1), we have that

$$\begin{aligned}(a\delta_s b\delta_t)c\delta_r &= \pi_s(\pi_{s^*}(a)b)\delta_{st}c\delta_r \\ &= \pi_{st}\pi_{(st)^*}[\pi_s(\pi_{s^*}(a)b)]c\delta_{str}.\end{aligned}$$

Observe that $\pi_s(\pi_{s^*}(a)b)$ is in $\pi_s(X_{s^*} \cap X_t) = X_s \cap X_{st}$ since

$$\pi_s(\pi_{s^*}(a)b) \in X_{s^*} \cap X_t.$$

Therefore, by Definition 1.1 part (*iii*) we have that

$$\begin{aligned}\pi_{(st)^*}[\pi_s(\pi_{s^*}(a)b)] &= \pi_{t^*}(\pi_{s^*}[\pi_s(\pi_{s^*}(a)b)]) \\ &= \pi_{t^*}(\pi_{s^*}(a)b).\end{aligned}$$

The last equality is obtained by Definition 1.1 part (i). By similar argument, we can split π_{st} which gives that the left hand side of (1) is equal to

$$\pi_s[\pi_t(\pi_{t^*}(\pi_{s^*}(a)b)c)]\delta_{str}. \quad (2)$$

Comparing (2) with

$$a\delta_s(b\delta_t c\delta_r) = a\delta_s\pi_t(\pi_{t^*}(b)c)\delta_{tr} = \pi_s[\pi_{s^*}(a)\pi_t(\pi_{t^*}(b)c)]\delta_{str}$$

and applying π_{s^*}, we deduce that (1) is valid if and only if

$$\pi_t\{\pi_{t^*}(\pi_{s^*}(a)b)c\} = \pi_{s^*}(a)\pi_t(\pi_{t^*}(b)c)$$

is verified for all $a \in X_s, b \in X_t$, and $c \in X_r$. Because $\pi_{s^*}: X_s \mapsto X_{s^*}$ is an isomorphism, $\pi_{s^*}(a)$ runs over X_{s^*} when a varies in X_s, consequently, the above required is equivalent to say that

$$\pi_t\{\pi_{t^*}(ab)c\} = a\pi_t(\pi_{t^*}(b)c) \quad (3)$$

for every $a \in X_{s^*}$, $b \in X_t$,and $c \in X_r$. If $s = r = 1$, then $X_s = X_r = \mathcal{A}$, and in this case $\mathcal{A} \rtimes_\pi S$ is associative if and only if (3) holds for arbitrary $t \in S$, $a, c \in \mathcal{A}$, and $b \in X_t$, but it is equivalent to the following:

$$(\pi_t \circ R_c \circ \pi_{t^*}) \circ L_a = L_a \circ (\pi_t \circ R_c \circ \pi_{t^*}). \quad (4)$$

If we consider R_c and L_a as a right and left multipliers of X_{t^*} respectively, then by [6, Proposition 2.7] $\pi_t \circ R_c \circ \pi_{t^*}$ is a right multiplier of X_t, thus the proof will be completed since each $X_t (t \in S)$ are supposed to be (L,R)-associative. □

As a consequence of Theorem 2.1, we have the following corollary.

Corollary 2.2. Let π be a partial action of a unital inverse semigroup S on an algebra $\mathcal{A}$ such that each X_t is an idempotent or non-degenerate ideal of $\mathcal{A}$ for each $t \in S$. Then $\mathcal{A} \rtimes_\pi S$ is associative.

Proof. This follows from [6, Proposition 2.5] and Theorem 2.1. □

We close this section by introducing the concept of *strongly associative* algebra and a consequence of it.

Definition 2.3. We shall say that an algebra $\mathcal{A}$ is strongly associative if for any unital inverse semigroup S and any partial action of S on $\mathcal{A}$, namely π, the skew inverse semigroup ring $\mathcal{A} \rtimes_\pi S$ is associative.

As a consequence of Corollary 2.2 and [1, Proposition 2.6] we have:

Corollary 2.4. A semiprime algebra $\mathcal{A}$ is strongly associative.

3. Morita Equivalence

Let S be an inverse semigroup, $\mathcal{B}$ be an algebra, and β be a semigroup homomorphism from S to Aut($\mathcal{B}$). In addition, suppose that $\mathcal{A}$ is an ideal of $\mathcal{B}$. For each $s \in S$ put $X_s = \beta_s(\mathcal{A}) \cap \mathcal{A}$ and $\pi_s = \beta_s|_{X_{s^*}}$. We now prove that $s \mapsto \pi_s$ is a partial action of S on $\mathcal{A}$. To do this, consider the following equation

$$\pi_s(X_{s^*}) = \beta_s(\beta_{s^*}(\mathcal{A}) \cap \mathcal{A}) = \beta_s(\beta_{s^*}(\mathcal{A})) \cap \beta_s(\mathcal{A}). \quad (5)$$

The last equality in (5) is obtained from the injectivity of β_s. Next, since for each $s \in S$, $\beta_s = \beta_{ss^*s} = \beta_s\beta_{s^*}\beta_s$, we have $\beta_{s^*} = \beta_s^{-1}$. So, the right hand side of (5) is equal to $\mathcal{A} \cap \beta_{s^*}(\mathcal{A})$, therefore, the rang of π_s is X_s. It is easy to verify that $\pi_s \circ \pi_{s^*}$ and $\pi_{s^*} \circ \pi_s$ are identity maps, hence, the part (i) of the Definition 1.1 is satisfied. Now, let us consider the part (ii) of Definition 1.1. Suppose that s, t are elements of S, then

$$\begin{aligned}\pi_s(X_{s^*} \cap X_t) &= \beta_s(\beta_{s^*}(\mathcal{A}) \cap \beta_t(\mathcal{A}) \cap \mathcal{A}) \\ &= \beta_s(\beta_{s^*}(\mathcal{A})) \cap \beta_s(\beta_t(\mathcal{A})) \cap \beta_s(\mathcal{A}) \\ &= \mathcal{A} \cap \beta_s \circ \beta_t(\mathcal{A}) \cap \beta_s(\mathcal{A}) \\ &= \mathcal{A} \cap \beta_{st}(\mathcal{A}) \cap \beta_s(\mathcal{A}) = X_s \cap X_{st}.\end{aligned}$$

If $x \in X_{t^*} \cap X_{(st)^*}$, then

$$\pi_t(x) \in \pi_t(X_{t^*} \cap X_{(st)^*}) = \pi_t(\pi_{t^*}(X_t \cap X_{s^*})) = X_t \cap X_{s^*} \subseteq X_{s^*},$$

so, the composition $\pi_s(\pi_t(x))$ makes sense. On the other hand,

$$\pi_s(\pi_t(x)) = \beta_s(\beta_t(x)) = \beta_{st}(x) = \pi_{st}(x).$$

Accordingly, the map $s \mapsto \pi_s$ is a partial action of S on $\mathcal{A}$ which is called the restriction of β to $\mathcal{A}$. If in addition $\mathcal{B}$ is generated by $\bigcup_{s\in S}\beta_s(\mathcal{A})$, then we say that π is an admissible restriction of β to $\mathcal{A}$.

Definition 3.1. Suppose that $\pi = \{\pi_s: X_{s^*} \mapsto X_s: s \in S\}$ and $\pi' = \{\pi'_s: X'_{s^*} \mapsto X'_s: s \in S\}$ are partial actions of S on algebras $\mathcal{A}$ and $\mathcal{A}'$

respectively. We will say that π and π' are equivalent if there exists an isomorphism

$\varphi: \mathcal{A} \mapsto \mathcal{A}'$ such that the following are satisfied:

(i) $\varphi(X_s) = X'_s$ for all $s \in S$,

(ii) $\pi'_s \circ \varphi(x) = \varphi \circ \pi_s(x)$ for all $x \in X_{s^*}$.

Here, we attempt to define an enveloping action for a partial action of an inverse semigroup S.

Definition 3.2. A semigroup homomorphism β from S to Aut($\mathcal{B}$), where $\mathcal{B}$ is an algebra, is said to be an enveloping action for a partial action π of S on an algebra $\mathcal{A}$ if π is equivalent to an admissible restriction of β to an ideal of $\mathcal{B}$.

Proposition 3.3. Suppose that S is a unital inverse semigroup, and β is an enveloping action for a partial action π of S on an algebra $\mathcal{A}$. Then the skew inverse semigroup ring $\mathcal{A} \rtimes_\pi S$ has an embeding into $\mathcal{B} \rtimes_\beta S$.

Proof. Since β is an enveloping action for π, there exists an algebra isomorphism φ of $\mathcal{A}$ onto an ideal of $\mathcal{B}$ such that for all $s \in S$ the followings are satisfied:

(i) $\varphi(X_s) = \varphi(\mathcal{A}) \cap \beta_s(\mathcal{A})$,

(ii) $\varphi \circ \pi_s = \beta_s \circ \varphi$.

Now, define

$$\tilde{\psi}: L \to \mathcal{B} \rtimes_\beta S$$

Clearly, $\tilde{\psi}$ vanishes on I, as a consequence, it can be extended to a map $\psi: \mathcal{A} \rtimes_\pi S \to \mathcal{B} \rtimes_\beta S$. Obviously ψ is injective, and the proof is complete. □

Now, we state the Morita equivalence in such a way that it is applicable in the sequel. We start with a few definitions and introduce notations and terminologies that is compatible to our structure. Throughout this section, we will let S be a unital inverse semigroup, β a semigroup homomorphism from S into Aut($\mathcal{B}$) which is enveloping action for a partial action π on a unital algebra $\mathcal{A}$. Set $R = \mathcal{A} \rtimes_\pi S$ and $R' = \mathcal{B} \rtimes_\beta S$. The main result of this section is to show that R and R' are Morita equivalence.

Definition 3.4. By a Morita context we mean a six-tuple $(R, R', M, M', \tau, \tau')$, where

(i) R and R' are rings,

(ii) M is an $R - R'$-bimodule,

(iii) M' is an $R' - R$-bimodule,

(iv) $\tau: M \otimes M' \mapsto R$ is a bimodule map,

(v) $\tau': M' \otimes M \mapsto R'$ is a bimodule map,

such that

$$\tau(x \otimes x')y = x\tau'(x' \otimes y) \; for\ all$$
$$x, y \in M, x' \in M',$$

and

$$\tau'(x' \otimes x)y = x'\tau(x \otimes y') \; for\ all$$
$$x', y' \in M, x \in M.$$

By Morita's fundamental results [2, Theorems 4.1.4 and 4.1.17], given a Morita context with τ and τ' onto the categories of R-modules and R'-modules are equivalent. In this case, R and R' are said to be Morita equivalent. However, The following propositions will lead us to the main result of this section.

Let $(\beta, \mathcal{B})$ be an enveloping action for a partial action $(\pi, \mathcal{A})$ (as mentioned above, we have supposed that both $\mathcal{B}$ and $\mathcal{A}$ are unital algebras). Consider two linear subspaces $M, N \subseteq \mathcal{B} \rtimes_\beta S$ given by

$$M = \{\textstyle\sum_{s\in S} \overline{c_s\delta_s} : c_s \in \mathcal{A} \text{ for all } s\},$$

and

$$N = \{\textstyle\sum_{s\in S} \overline{c_s\delta_s} : c_s \in \beta_s(\mathcal{A}) \text{ for all } s\}.$$

Our main Theorem 3.8 is the consequence of the three following Propositions.

With respect to subspaces we have the following Proposition.

Proposition 3.5. Let M and N be as mentioned above. Then M is a right ideal and N is a left ideal of $\mathcal{B} \rtimes_\beta S$.

Proof. Suppose that $\overline{c\delta_s}$ is an element of M, where $s \in S$ and $c \in \mathcal{A}$. To prove that M is a right ideal of $\mathcal{B} \rtimes_\beta S$, we shall show that $\overline{b\delta_t}.\overline{c\delta_s}$ is in M, where $t \in S$ and $b \in \mathcal{B}$. For such element, we have

$$\overline{c\delta_s}.\overline{b\delta_t} = \overline{\beta_s(\beta_{s^*}(c)b)\delta_{st}} = \overline{c\beta_s(b)\delta_{st}},$$

but $\mathcal{A}$ is an ideal of $\mathcal{B}$, so, $c\beta_s(b) \in \mathcal{A}$. As a result, $\overline{c\delta_s}.\overline{b\delta_t}$ is in M.

Let $\overline{c\delta_s}$ be in N, where $s \in S$ and $c = \beta_s(c')$ with c' in $\mathcal{A}$. Now, the conclusion follows if we can establish that $\overline{b\delta_t}.\overline{c\delta_s}$ is in N, where $t \in S$ and $b \in \mathcal{B}$. Consider

$$\overline{b\delta_t}.\overline{c\delta_s} = \overline{\beta_t(\beta_{t^*}(b)c)}\delta_{st} = \overline{b\beta_{ts}(c')\delta_{ts}}.$$

The fact that $\beta_s(\mathcal{A})$ is an ideal of $\mathcal{B}$ implies that $b\beta_{ts} \in \mathcal{A}$, consequently, N is a left ideal of $\mathcal{B} \rtimes_\beta S$.□

The next Proposition gives us more information about M and N.

Proposition 3.6. Considering the linear subspaces M, N, we have that M is a left $\mathcal{A} \rtimes_\pi S$-module and N is a right $\mathcal{A} \rtimes_\pi S$-module.

Proof. Suppose that $\overline{c\delta_s}$ is an element of M, where $s \in S$ and $c \in \mathcal{A}$, and let $\overline{a\delta_t} \in \mathcal{A} \rtimes_\pi S$ where $t \in S$ and $a \in X_t$. Then

$$\overline{a\delta_t}.\overline{c\delta_s} = \overline{a\beta_t(c)\delta_{ts}}.$$

Since $\mathcal{A}$ is an ideal of $\mathcal{B}$, $a\beta_t(c) \in \mathcal{A}$, as a result, $\overline{a\beta_t(c)\delta_{ts}}$ is in M. Hence, M is a left $\mathcal{A} \rtimes_\pi S$-module. For the rest of the proof, take $\overline{c\delta_s}$ in N and $\overline{a\delta_t}$ in $\mathcal{A} \rtimes_\pi S$, where $s, t \in S$, $a \in X_t$, and $c = \beta_s(c')$ for some $c' \in \mathcal{A}$. Then

$$\begin{aligned}\overline{c\delta_s}.\overline{a\delta_t} &= \overline{\beta_s(\beta_{s^*}(c)a)\delta_{st}}\\ &= \overline{\beta_s(c')\beta_s(a)\delta_{st}}\\ &= \overline{\beta_s(c'a)\delta_{st}}.\end{aligned}$$

Since X_t is an ideal of $\mathcal{A}$, $c'a \in X_t$, therefore, $c'a = \pi_t(x)$ for some x in X_{t^*}. Thus

$$\overline{\beta_s(c'a)\delta_{st}} = \overline{\beta_s(\pi_t(x))\delta_{st}} = \overline{\beta_s\beta_t(x)\delta_{st}}.$$

The last above equality is obtained from the definition of enveloping action. So, N is a right $\mathcal{A} \times_\pi S$-module. □

The next Proposition introduces an idea that will lead us to the main Theorem of this section. Before we state the Proposition, we need to introduce a convention. Given two linear spaces X, Y, we will denote by XY the linear span of the set of products xy with $x \in X$ and $y \in Y$. We use this notation without any comment.

Proposition 3.7. Let S be a unital inverse semigroup. Then $MN = \mathcal{A} \rtimes_\pi S$ and $NM = \mathcal{B} \rtimes_\beta S$.

Proof. Let $\overline{c\delta_s} \in M$, $\overline{c'\delta_t} \in N$ where $s, t \in S$, $c \in \mathcal{A}$, and $c' = \beta_t(c'')$ with $c'' \in \mathcal{A}$. Observe that

$$\overline{c\delta_s}.\overline{c'\delta_t} = \overline{c\beta_s(c')\delta_{st}} = \overline{c\beta_{st}(c'')\delta_{st}} \in \mathcal{A} \rtimes_\pi S.$$

Suppose that $\overline{c\delta_t} \in \mathcal{A} \rtimes_\pi S$, indeed, $\overline{c\delta_t} \in N$ because $c \in X_t = \pi_t(X_{t^*}) \subseteq \beta_t(\mathcal{A})$. Letting $1_\mathcal{A}$ be the unit element of $\mathcal{A}$, we have that $1_\mathcal{A}\delta_1 \in M$ and

$$1_\mathcal{A}\delta_1.c\delta_t = 1_\mathcal{A}c\delta_t = c\delta_t.$$

As a matter of fact, the above equation implies that $\mathcal{A} \rtimes_\pi S \subseteq MN$, as a result of the last computation we have $\mathcal{A} \rtimes_\pi S = MN$. Because M, N are respectively right and left ideals of $\mathcal{B} \rtimes_\beta S$, by Proposition 5, one has that $NM \subseteq \mathcal{B} \rtimes_\beta S$. Now, consider an element $\beta_s(c)$, where $s \in S$ and $c \in \mathcal{A}$. clearly, $1_\mathcal{A}\beta_s(c)$ is an element of $\mathcal{A}$, therefore, $\overline{1_\mathcal{A}\beta_s(c)\delta_t} \in M$ with $t \in S$. Additionally, we have $\overline{1_\mathcal{A}\delta_1} \in N$ since $\beta_1(1_\mathcal{A}) = 1_\mathcal{A}$. The equation $\overline{1_\mathcal{A}\delta_1}.\overline{1_\mathcal{A}\beta_s(c)\delta_t} = \overline{\beta_s(c)\delta_t}$ and the assumption that $\mathcal{B}$ is generated by $\cup_{s\in S}\beta_s(\mathcal{A})$ implies that $\mathcal{B}\delta_t \subseteq NM$, consequently, $\mathcal{B} \rtimes_\beta S \subseteq NM$, thus, $\mathcal{B} \rtimes_\beta S = NM$ □

Now, we are ready to give a Morita context. So far, we have showed that M is a $\mathcal{A} \rtimes_\pi S$-$\mathcal{B} \rtimes_\beta S$-bimodule and N is a $\mathcal{B} \rtimes_\beta S$-$\mathcal{A} \rtimes_\pi S$-bimodule. Consider two maps

$$\tau: M \otimes_{\mathcal{B}\rtimes_\beta S} N \mapsto \mathcal{A} \rtimes_\pi S$$
$$m \otimes n \mapsto mn,$$

and

$$\tau' : N \otimes_{\mathcal{A}\rtimes_\pi S} M \mapsto \mathcal{B} \rtimes_\beta S$$

One easily can verify that τ, τ' are bimodule maps satisfying the conditions of Definition 3.4, in fact, the six-tuple $(\mathcal{A} \rtimes_\pi S, \mathcal{B} \rtimes_\beta S, M, N, \tau, \tau')$ is a Morita context with τ, τ' onto by Proposition 3.7.

We close this section with the following main Theorem.

Theorem 3.8. Let S be a unital inverse semigroup, and β be a global action of S on a unital algebra $\mathcal{B}$ which is enveloping for a partial action π of S on a unital algebra $\mathcal{A}$. Then $\mathcal{A} \rtimes_\pi S$ and $\mathcal{B} \rtimes_\beta S$ are Morita equivalent.

References

[1] R. Exel, Circle Actions on C^*-algebras, partial automorphisms and generalized pimsner-voiculescu exact sequences, J. Funct. Anal 122 (1994) 361-401.

[2] K. McClanahan, K-Theory for partial crossed products by discrete groups, J. Funct. Anal 130 (1995) 77-117.

[3] A. Buss, R. Exel, Inverse semigroup expansions and their actions on C^*-algebras, J. Illinois. Math. 56 (2014) 1185-1212.

[4] J.M.G. Fell, P.S. Doran, Representations of $*$-Algebras, Locally Compact Groups, and Banach $*$-Algebraic Bundles II, Academic Press INC, London, 1988.

[5] G. J. Murphy, C^*- Algebras and Operator Theory, Academic press, INC, 1990.

[6] M. Dokuchaev, R. Exel, Associativity of crossed products by partial actions, enveloping actions and partial representations, Trans. Amer. Math. Soc. 357 (5) (2004) 1931-1952.