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The Level Curves and Surfaces on Time Scales

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Abstract: The general idea of this paper is to study level curves and surfaces by considering delta gradient functions on time scales. Aided by the definition of the delta gradient function, some geometric structures of level curves and surfaces are investigated.

Keywords: Time scales, delta-gradient function, level surface, normal vector, delta calculus.

1. Introduction

Calculus of time scales was first introduced by Stefan Hilger in his 1988 Ph.D. dissertation in order to create a theory that could unify both discrete and continuous analyses. Time-scale calculus has recently received a lot of attention and has proved to be useful in many fields of science that involve dynamical systems [1]. Bohner & Guseinov published a paper regarding the partial differentiation of time scales [2], while the theory of dynamical systems on time scales was developed by the editors, Bohner & Peterson [3-4]. The directional nabla derivative and curves on $n-$dimensional time scales Λ^n are presented in [5], while the delta derivative of a vector field along a curve is given in [6]. The surfaces parameterized by the product of two arbitrary time scales are studied in [7]. The normal and osculating planes of curves parameterized by a compact subinterval of a time scale are presented in [8]. A generalization of the notion of a regular curve was introduced in [9], and some generalized geometric structures are introduced in [10-11].

The gradient is a generalization of the familiar concept of the derivative of a function. More precisely, the gradient points in the direction of the greatest rate of increase of the function, and its magnitude is the slope of the function in that direction.

In this paper, we will look for a new concept regarding a time scale. First, we will define the delta gradient function and then calculate some of its properties on time scales. The delta gradient function proves to be a useful tool that helps unify discrete and continuous functions. Next, we present the basic definitions and theorems for constructing a level curve and surface using the delta gradient function.

2. Preliminaries

Let $n \in N$ be fixed. Further, for each $i \in \{1,2,\ldots\ldots,n\}$ let $\mathbb{T}_i$ denote a time scale, that is, $\mathbb{T}_i$ is a nonempty closed subset of the real number $\mathbb{R}$. Let us set

$$\Lambda^n = \mathbb{T}_1 \times \ldots \times \mathbb{T}_n = \{t = (t_1, \ldots, t_n) : t_i \in \mathbb{T}_i \; for \; all \; i \in \{1, \ldots\ldots, n\}\}.$$

We call Λ^n an $n-$ dimensional time scale. The set Λ^n is a complete metric space where the metric d is defined by

$$d(t,s) = \left(\sum_{i=1}^{n} | t_i - s_i|^2\right)^{\frac{1}{2}} \quad for \;\; t,s \in \Lambda^n.$$

Let σ_i and ρ_i denote, respectively, the forward- and backward-jump operators in $\mathbb{T}_i$. Remember that for $u \in \mathbb{T}_i$, the forward-jump operator $\sigma_i : \mathbb{T}_i \longrightarrow \mathbb{T}_i$ is defined by

$$\sigma_i(u) = \inf\{ v \in \mathbb{T}_i : v > u\}.$$

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In this definition we put $\sigma_i(\max \mathbb{T}_i) = \max \mathbb{T}_i$, if $\mathbb{T}_i$ has a finite maximum. If $\sigma_i > u$, then we say that u is right-scattered in $\mathbb{T}_i$ and if $\sigma_i(u) = u$, then u is called right-dense in $\mathbb{T}_i$. If $\mathbb{T}_i$ has a right-scattered minimum m, then we define $(\mathbb{T}_i)^k = \mathbb{T}_i \backslash \{m\}$, otherwise $(\mathbb{T}_i)^k = \mathbb{T}_i$.

Theorem 2.1. If the time scale $\mathbb{T}_i$ is $\mathbb{R}$, then the forward-jump operator becomes $\sigma_i(t) = t$. If the time scale $\mathbb{T}_i$ is $\mathbb{Z}$, then the forward-jump operator becomes $\sigma_i(t) = t + 1$.

Definition 2.1. If $f: \mathbb{T} \to \mathbb{R}$ is a function and $t \in \mathbb{T}^k$, then the delta derivative of f at the point t is defined to be the number $f^\Delta(t)$, which has the property that for each $\varepsilon > 0$ there is a neighborhood U of t such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s|$$

for all $s \in U$.

Theorem 2.2. For $f: \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$ the following holds

(i) If f is $\Delta-$ differentiable at t, then f is continuous at t.

(ii) If f is continuous at t and t is right-scattered, then f is $\Delta-$differentiable at t and

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}.$$

(iii) If t is right-dense, then f is $\Delta-$differentiable at t if and only if the limit

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exist as a finite number. In this case $f^\Delta(t)$ is equal to this limit.

(iv) If f is $\Delta-$differentiable at t, then

$$f(\sigma(t)) = f(t) + [\sigma(t) - t] f^\Delta(t).$$

Theorem 2.3. If $f, g: \mathbb{T} \to \mathbb{R}$ are $\Delta-$differentiable at $t \in \mathbb{T}^k$, then

(i) $f + g$ is $\Delta-$differentiable at t and

$$(f + g)^\Delta(t) = f^\Delta(t) + g^\Delta(t)$$

(ii) For any constant c, cf is $\Delta-$differentiable at t and

$$(cf)^\Delta(t) = c.f^\Delta(t)$$

(iii) $f.g$ is $\Delta-$differentiable at t and

$$(f.g)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t)$$
$$= f(t)g^\Delta(t) + f^\Delta(t)g(\sigma(t))$$

(iv) If $g(t)g(\sigma(t)) \neq 0$,

then $\frac{f}{g}$ is $\Delta-$differentiable at t and

$$(\frac{f}{g})^\Delta(t) = \frac{f^\Delta(t)g(t) - g^\Delta(t)f(t)}{g(t)g(\sigma(t))}$$

Definition 2.2. Let the function $f: \mathbb{T}_1 \times \ldots \times \mathbb{T}_n \longrightarrow \mathbb{R}$ be given. The directional nabla derivative of the function f at the point $(t_1^0, \ldots, t_n^0)$ in the direction of the vector w (along w) is defined as the number

$$\frac{\partial f(t_1^0, \ldots, t_n^0)}{\nabla w} = F^\nabla(0)$$

provided it exists, where

$$F(\xi) = f(t_1^0 + \xi w_1, \ldots, t_n^0 + \xi w_n) \quad for \xi \in \mathbb{T}.$$

Definition 2.3. Let the function $f: \mathbb{T}_1 \times \ldots \times \mathbb{T}_n \longrightarrow \mathbb{R}$ be given. The directional delta derivative of the function f at the point $(t_1^0, \ldots, t_n^0)$ in the direction of the vector w (along w) is defined as the number

$$\frac{\partial f(t_1^0, \ldots, t_n^0)}{\Delta w} = F^\Delta(0)$$

provided it exists, where

$$F(\xi) = f(t_1^0 + \xi w_1, \ldots, t_n^0 + \xi w_n) \quad for \xi \in \mathbb{T}.$$

3. Main Results

Let $x_i: \Lambda^n \to \mathbb{T}_i$ be a Euclidean coordinate function on a time scale for all $1 \leq i \leq n$, $n \in \mathbb{N}$ denoted by the set $\{x_1, x_2, \ldots, x_n\}$. Let $f: \Lambda^n \to \Lambda^n$ be a function described by $f(P) = (f_1(P), f_2(P), \ldots, f_n(P))$ at a point$P \in \Lambda^n$. The function f is called a σ_1-completely delta differentiable function at point P provided that all f_i, $i = 1,2, \ldots, n$ functions are σ_1- completely delta differentiable at point P. All functions of this type are denoted by $C^\Delta_{\sigma_1}$. Let $\chi(\Lambda^n)$ be a set of σ_1-completely delta differentiable vector fields.

Definition 3.1. If the function f is σ_1- completely delta differentiable at point $P(t_1^0, t_2^0, \ldots, t_n^0)$, then **the delta gradient function** is defined as a map

$$\nabla_\Delta : C^\Delta_{\sigma_1} \to \chi(\Lambda^n)$$

$$f \to \nabla_\Delta f = \sum_{i=1}^{n} \frac{\partial f}{\Delta_i x_i}\frac{\partial}{\Delta_i x_i}$$

where

$$\{\frac{\partial}{\Delta_1 x_1}, \frac{\partial}{\Delta_2 x_2}, \dots, \frac{\partial}{\Delta_n x_n}\}$$

is the basis for $\chi(\Lambda^n)$.

Definition 3.2. If the delta gradient is defined by $\nabla_\Delta f$, then **the delta gradient operator** is denoted by ∇_Δ, and is expressed by the formula

$$\nabla_\Delta = \sum_{i=1}^{n} \frac{\partial}{\Delta_i x_i}\frac{\partial}{\Delta_i x_i}.$$

Theorem 3.1. Let $c \in \mathbb{R}$, $f, g \in C^\Delta_{\sigma_1}$. Then the following properties are proven for a delta-gradient operator.

(i) $\nabla_\Delta(f+g) = \nabla_\Delta f + \nabla_\Delta g$

(ii) $\nabla_\Delta(c.f) = c.\nabla_\Delta f$

(iii) $\nabla_\Delta(f.g) = g.\nabla_\Delta f + f^{\sigma_1}\nabla_\Delta g$

(iv) $\nabla_\Delta(\frac{f}{g}) = \frac{g.\nabla_\Delta f - f.\nabla_\Delta g}{g.g^{\sigma_1}}$

Proof: Considering definition (2.3), we obtain the proofs of the theorem as follows

$$(\boldsymbol{i}) \quad \nabla_\Delta(f+g) = \sum_{i=1}^{n} \frac{\partial(f+g)}{\Delta_i x_i}\frac{\partial}{\Delta_i x_i}$$

$$= \sum_{i=1}^{n} \frac{\partial f}{\Delta_i x_i}\frac{\partial}{\Delta_i x_i} + \sum_{i=1}^{n} \frac{\partial g}{\Delta_i x_i}\frac{\partial}{\Delta_i x_i}$$

$$= \nabla_\Delta f + \nabla_\Delta g$$

$$(ii) \quad \nabla_\Delta(cf) = \sum_{i=1}^{n} \frac{\partial(cf)}{\Delta_i x_i}\frac{\partial}{\Delta_i x_i}$$

$$= c\sum_{i=1}^{n} \frac{\partial f}{\Delta_i x_i}\frac{\partial}{\Delta_i x_i}$$

$$= c\nabla_\Delta f$$

$$(iii) \quad \nabla_\Delta(f.g) = \sum_{i=1}^{n} \frac{\partial(f.g)}{\Delta_i x_i}\frac{\partial}{\Delta_i x_i}$$

$$= \sum_{i=1}^{n} \left[\frac{\partial f}{\Delta_i x_i} g + f(\sigma_1(t))\frac{\partial g}{\Delta_i x_i}\right]\frac{\partial}{\Delta_i x_i}$$

$$= \sum_{i=1}^{n} \frac{\partial f}{\Delta_i x_i} g \frac{\partial}{\Delta_i x_i} + \sum_{i=1}^{n} f\left(\sigma_1(t)\right)\frac{\partial g}{\Delta_i x_i}\frac{\partial}{\Delta_i x_i}$$

$$= g\sum_{i=1}^{n} \frac{\partial f}{\Delta_i x_i}\frac{\partial}{\Delta_i x_i} + f\left(\sigma_1(t)\right)\sum_{i=1}^{n} \frac{\partial g}{\Delta_i x_i}\frac{\partial}{\Delta_i x_i}$$

$$= g\nabla_\Delta f + f^{\sigma_1}\nabla_\Delta g$$

$$(iv) \quad \nabla_\Delta\left(\frac{f}{g}\right) = \sum_{i=1}^{n} \frac{\partial\left(\frac{f}{g}\right)}{\Delta_i x_i}\frac{\partial}{\Delta_i x_i}$$

$$= \frac{1}{g.g(\sigma_1(t))}\sum_{i=1}^{n} \left[g.\frac{\partial f}{\Delta_i x_i} - f.\frac{\partial f}{\Delta_i x_i}\right]\frac{\partial}{\Delta_i x_i}$$

$$= \frac{1}{g.g^{\sigma_1}}\left[g.\sum_{i=1}^{n} \frac{\partial f}{\Delta_i x_i}\frac{\partial}{\Delta_i x_i} - f.\sum_{i=1}^{n} \frac{\partial f}{\Delta_i x_i}\frac{\partial}{\Delta_i x_i}\right]$$

$$= \frac{1}{g.g^{\sigma_1}}[g.\nabla_\Delta f - f\nabla_\Delta g]$$

$$= \frac{g.\nabla_\Delta f - f\nabla_\Delta g}{g.g^{\sigma_1}}$$

Theorem 3.2. The delta-gradient operator is a linear mapping.

Proof: Let $a, b \in \mathbb{R}$ and $f, g \in C{\sigma_1}^\Delta$. Then by theorem(1), we have

$$\nabla_\Delta(af+bg) = \sum_{i=1}^{n} \frac{\partial(af+bg)}{\Delta_i x_i}\frac{\partial}{\Delta_i x_i}$$

$$= \sum_{i=1}^{n} \frac{\partial(af)}{\Delta_i x_i}\frac{\partial}{\Delta_i x_i} + \sum_{i=1}^{n} \frac{\partial(bg)}{\Delta_i x_i}\frac{\partial}{\Delta_i x_i}$$

$$= a\sum_{i=1}^{n} \frac{\partial f}{\Delta_i x_i}\frac{\partial}{\Delta_i x_i} + b\sum_{i=1}^{n} \frac{\partial g}{\Delta_i x_i}\frac{\partial}{\Delta_i x_i}$$

$$= a\nabla_\Delta f + b\nabla_\Delta g$$

Thus the proof is completed.

Definition 3.3. Let $f(x_1, x_2)$ be a delta differentiable scalar function at point (a, b) in Λ^2. The equation

$f(x_1, x_2) = k$ is called a '**level curve on time scales**', for any constant $k \in \mathbb{R}$.

Theorem 3. Let f be a delta differentiable scalar function on Λ^2. If $f(x_1, x_2) = k$ and its $\nabla_\Delta f$ delta-gradient are not equal to zero at point (a, b), then the delta-gradient is orthogonal to the level curve at (a, b).

Proof: Considering the definition of the closed function theorem, we obtain the following equation:

$$f(x_1, x_2) - f(a, b) = 0.$$

The tangent-line equation of the level curve is

$$\frac{\partial f}{\Delta_1 x_1}(x_1 - a) + \frac{\partial f}{\Delta_2 x_2}(x_2 - b) = 0$$

where (a, b) is the point of the line. Let (x_1^0, x_2^0) be another point on the tangent line. Hence, the normal vector of the tangent line is expressed by the formula

$$u = (x_1^0 - a)\,\vec{i} + (x_2^0 - b)\,\vec{j}.$$

Therefore we obtain the following equations:

$$u.\nabla_\Delta f =$$

$$((x_1^0 - a)\,\vec{i} + (x_2^0 - b)\,\vec{j})(\frac{\partial f}{\Delta_1 x_1}\,\vec{i} + \frac{\partial f}{\Delta_2 x_2}\,\vec{j})$$

$$= \frac{\partial f}{\Delta_1 x_1}(x_1 - a) + \frac{\partial f}{\Delta_2 x_2}(x_2 - b) = 0.$$

Thus we have proved that $\vec{u}$ is perpendicular to $\nabla_\Delta f$.

Definition 3.4. Let $f(x_1, x_2, x_3)$ be a delta differentiable scalar function at point (a, b, c) in Λ^3. The equation

$$f(x_1, x_2, x_3) = k$$

is called a "**level surface on time scales**", for any constant $k \in \mathbb{R}$.

Theorem 3.4. Let f be a delta differentiable scalar function on Λ^3. The delta-gradient of the function f is perpendicular to the level surface at point $P_0(a, b, c) \in \Lambda^3$.

Proof: Taking the differential of both sides of this equality we obtain the following equation:

$$\frac{df}{\Delta t} = \frac{d(k)}{\Delta t}$$

$$\sum_{i=1}^{n} \frac{df}{\Delta_i x_i}\frac{dx}{\Delta_i t} = 0$$

$$\nabla_\Delta f.(\frac{dx_1}{\Delta_i t}, \dots, \frac{dx_n}{\Delta_i t}) = 0$$

$$\nabla_\Delta f.\frac{d\,\vec{r}}{\Delta t} = 0$$

where $\frac{d\vec{r}}{\Delta t}$ is the tangent vector of the level surface at P_0. Therefore $\nabla_\Delta f$ is perpendicular to the level surface at P_0.

Remark 3.1. It is not difficult to see that the normal vector of the level surface at point P is $\vec{N} = \nabla_\Delta f(P)$. Therefore the unit normal vector of the level surface at point P is given by the equation

$$\vec{n} = \frac{\vec{N}}{\|\vec{N}\|} = \frac{\nabla_\Delta f(P)}{\|\nabla_\Delta f(P)\|}.$$

Remark 3.2. The **tangent plane** of an S level surface at point $P_0 \in \Lambda^3$ is denoted by

$$(R - R_0).\nabla_\Delta f(P_0) = 0.$$

Here $\vec{R}_0$ is a position vector of $P_0(a, b, c)$ and, R is a position vector of $P(x_1, x_2, x_3)$ where the point is on the tangential plane.

Remark 3.3. If $\nabla_\Delta f|_P = 0$, then point P is called a singular point, otherwise point P is called a regular point.

Theorem 3.5. If f is a delta differentiable function in x_1 and x_2, then f has a delta derivative in the direction of any unit vector $\vec{u} = (\vec{a}, \vec{b})$ and

$$\frac{\partial f(x_1, x_2)}{\Delta\,\vec{u}} = \nabla_\Delta f(x_1, x_2).\vec{u}$$

Proof: From the definition of the directional delta derivative (3)

$$\frac{\partial f(x_1^0, x_2^0)}{\Delta\,\vec{u}} = \frac{\partial f(x_1^0, x_2^0)}{\Delta x_1}\vec{a} + \frac{\partial f(x_1^0, x_2^0)}{\Delta x_2}\vec{b}$$

$$= < (\frac{\partial f(x_1^0, x_2^0)}{\Delta x_1}, \frac{\partial f(x_1^0, x_2^0)}{\Delta x_2}), (\vec{a}, \vec{b}) >$$

$$= \nabla_\Delta f(x_1^0, x_2^0).\vec{u}$$

Theorem 3.6. Suppose that f is a delta differentiable function and $\vec{u}$ is a unit vector. The maximum value of

$$\frac{\partial f(x_1, x_2)}{\Delta\, \vec{u}}$$

at a given point is $\| \nabla_\Delta f \|$, which occurs when $\vec{u}$ has the same direction as $\nabla_\Delta f$ at the given point.

Proof: We have already proved that

$$\frac{\partial f}{\Delta\, \vec{u}} = \nabla_\Delta f . \vec{u} = \| \nabla_\Delta f \| . \| \vec{u} \| . cos\theta$$

where θ is the smallest angle between $\vec{u}$ and $\overrightarrow{\nabla f}$. Since $\vec{u}$ is a unit vector, we have

$$\frac{\partial f}{\Delta\, \vec{u}} = \| \nabla_\Delta f \| . cos\theta.$$

So, we see that this is a maximum when $cos\theta$ is a maximum, i.e. when

$$cos\theta = 1.$$

In this case, we have

$$\frac{\partial f}{\Delta\, \vec{u}} = \| \nabla_\Delta f \|.$$

Since this occurs when $cos\theta = 1$, i.e. when $\theta = 0$, then $\vec{u}$ has the same direction as $\nabla_\Delta f$.

Remark 3.4. Using a similar argument, it is seen that the minimum value of $\partial f / \Delta\, \vec{u}$ at a given point is $-\| \nabla_\Delta f \|$ and it occurs when $\vec{u}$ has the direction of $-\nabla_\Delta f$ at the given point.

A Numerical Example. For example, let $f = 2x_1^2 - 3x_2^2$ be a level surface. Here, for the level surface (i.e. the unit normal vector), the tangent plane and the maximum value at point $P(1,-1)$ can be computed. Additionally, the delta derivative in the direction of the unit vector $\vec{u} = (1,-1)$ on time scales $\Lambda = \mathbb{T}_1 \times \mathbb{T}_2$ can be obtained by taking the time scale via the following conditions:

a) $\Lambda_1 = \mathbb{R} \times \mathbb{R}$,

b) $\Lambda_2 = \mathbb{Z} \times \mathbb{Z}$,

c) $\Lambda_3 = \mathbb{Z} \times \mathbb{R}$

By considering definition (3.1), the delta gradient of the level surface is obtained by

$$\nabla_\Delta f = \nabla_\Delta (2x_1^2 - 3x_2^2)$$

$$= \frac{\partial(2x_1^2 - 3x_2^2)}{\Delta_1 x_1} \frac{\partial}{\Delta_1 x_1} + \frac{\partial(2x_1^2 - 3x_2^2)}{\Delta_2 x_2} \frac{\partial}{\Delta_2 x_2}$$

$$= 2\frac{\partial(x_1^2)}{\Delta_1 x_1} \frac{\partial}{\Delta_1 x_1} - 3\frac{\partial(x_2^2)}{\Delta_2 x_2} \frac{\partial}{\Delta_2 x_2}$$

$$= 2.[\sigma_1(x_1) + x_1] \frac{\partial}{\Delta_1 x_1} - 3[\sigma_2(x_2) + x_2] \frac{\partial}{\Delta_2 x_2}$$

$$= (2[\sigma_1(x_1) + x_1], -3[\sigma_2(x_2) + x_2])$$

The normal to the delta gradient is

$$\| \nabla_\Delta f \| = \sqrt{(2[\sigma_1(x_1) + x_1])^2 + (-3[\sigma_2(x_2) + x_2])^2}.$$

It is not difficult to see that the unit normal vector of the level surface is

$$\vec{n} = \frac{\nabla_\Delta f}{\| \nabla_\Delta f \|}$$

$$= \frac{(2[\sigma_1(x_1) + x_1], -3[\sigma_2(x_2) + x_2])}{\sqrt{(2[\sigma_1(x_1) + x_1])^2 + (-3[\sigma_2(x_2) + x_2])^2}}$$

from remark (3.1). The delta derivative in the direction of any unit vector $\vec{u} = (1,-1)$ is given by the following equation

$$\frac{\partial f}{\Delta\, \vec{u}} = \nabla_\Delta f . \vec{u} = (2[\sigma_1(x_1) + x_1], -3[\sigma_2(x_2) + x_2]).(1,-1)$$

see theorem (2.3). The maximum value of $\partial f / \Delta\, \vec{u}$ at point $P(1,-1)$ is $\| \nabla_\Delta f(P) \|$. The plane tangential to the level surface f at point $P(1,-1)$ is calculated by

$$(\vec{R} - \vec{R}_0). \nabla_\Delta f(P_0) = 0.$$

Now, the above equations were considered for different time scales as $\Lambda_1 = \mathbb{R} \times \mathbb{R}$, $\Lambda_2 = \mathbb{Z} \times \mathbb{Z}$ and $\Lambda_3 = \mathbb{Z} \times \mathbb{R}$.

a) By taking the time scales as $\Lambda_1 = \mathbb{R} \times \mathbb{R}$, the forward-jump operators become $\sigma_1(t) = \sigma_2(t) = t$ from theorem(1). The delta gradient of the level surface of Λ_1 is given by the following equations, see Figure1.

$$\nabla_\Delta f = (2[\sigma_1(x_1) + x_1], -3[\sigma_2(x_2) + x_2])$$
$$= (2[x_1 + x_1], -3[x_2 + x_2])$$
$$= (4x_1, -6x_2)$$

It is easy to see that

$$\nabla_\Delta f = (4x_1, -6x_2)|_{P(1,-1)} = (4,6) \neq 0,$$

thus this point is a regular point on the level surface. The normal of the gradient vector is $\| \nabla_\Delta f \| = \sqrt{4^2 + 6^2} = 2\sqrt{13}$. By taking the Λ_1 time scale, the delta derivative in the direction of any unit vector $\vec{u} = (1,-1)$ is

$$\frac{\partial f}{\Delta \vec{u}} = \nabla_\Delta f . \vec{u}$$
$$= < (4x_1, -6x_2), (1,-1) >$$
$$= 4x_1 + 6x_2.$$

Thus, the maximum value of $\partial f / \Delta \vec{u}$ at point $P(1,-1)$ is $\| \nabla_\Delta f(P) \| = 2\sqrt{13}$. Next, the unit normal vector of the level surface is obtained via the following equation

$$\vec{n} = \frac{\nabla_\Delta f}{\| \nabla_\Delta f \|} = \frac{(4,6)}{2\sqrt{13}} = (\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}})$$

From remark (3.1), the following equation gives the tangential plane of the level surface at point $P(1,-1)$

$$(\vec{R} - \vec{R}_0). \nabla_\Delta f(P_0) = 0$$
$$[(x_1, x_2) - (1,-1)]. (4,6) = 0$$
$$(x_1 - 1, x_2 + 1). (4,6) = 0$$
$$2(x_1 - 1) + 3(x_2 + 1) = 0$$
$$2x_1 + 3x_2 + 1 = 0$$

b) The forward-jump operators of the time scale $\Lambda_2 = \mathbb{Z} \times \mathbb{Z}$ are $\sigma_1(t) = \sigma_2(t) = t + 1$. Thus, the delta gradient of the level surface Λ_2 is the following, see Figure 2.

$$\nabla_\Delta f = (2[\sigma_1(x_1) + x_1], -3[\sigma_2(x_2) + x_2])$$
$$= (2[x_1 + 1 + x_1], -3[x_2 + 1 + x_2])$$
$$= (4x_1 + 2, -6x_2 - 3)$$

We see that equation

$$\nabla_\Delta f = (4x_1 + 2, -6x_2 - 3)|_{P(1,-1)} = (6,-9) \neq 0$$

is obtained, thus this point is a regular point on the level surface.

The normal of the gradient vector is $\| \nabla_\Delta f \| = \sqrt{6^2 + (-9)^2} = \sqrt{117}$.

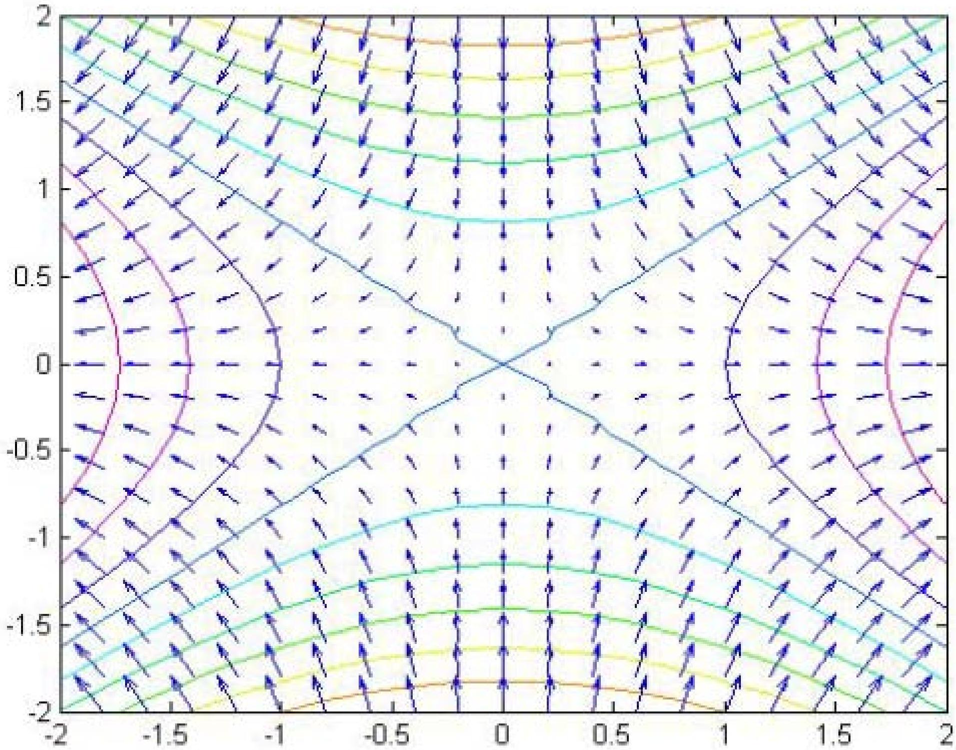

Fig. 1 The gradient vector field $\nabla_\Delta f$ on $\Lambda_1 = \mathbb{R} \times \mathbb{R}$.

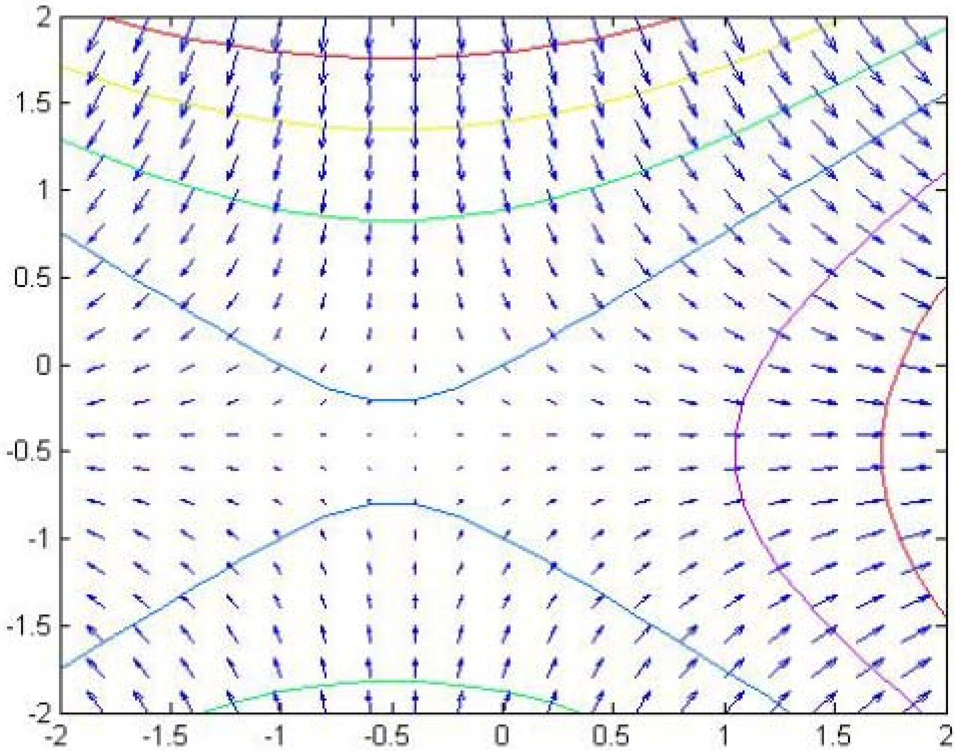

Fig. 2 The gradient vector field $\nabla_\Delta \mathrm{f}$ on $\Lambda_2 = \mathbb{Z} \times \mathbb{Z}$.

The delta derivative in the direction of the unit vector $\vec{u} = (1, -1)$ is

$$\frac{\partial f}{\Delta \vec{u}} = \nabla_\Delta f . \vec{u}$$
$$= < (4x_1 + 2, -6x_2 - 3), (1, -1) >$$
$$= 4x_1 + 6x_2 + 5.$$

Thus, the maximum value of $\partial f / \Delta \vec{u}$ at point $P(1, -1)$ is $\| \nabla_\Delta f(P) \| = \sqrt{117}$. Next, the unit normal vector of the level surface at point $P(1, -1)$ is

$$\vec{n} = \frac{\nabla_\Delta f}{\| \nabla_\Delta f \|} = \frac{(6, -9)}{\sqrt{117}} = (\frac{6}{\sqrt{117}}, \frac{-9}{\sqrt{117}}).$$

The tangential plane of the level surface at point $P(1, -1)$ on time scale Λ_2 is

$$(\vec{R} - \vec{R}_0) . \nabla_\Delta f(P_0) = 0$$
$$[(x_1, x_2) - (1, -1)] . (6, -9) = 0$$
$$(x_1 - 1, x_2 + 1) . (2, -3) = 0$$
$$2(x_1 - 1) - 3(x_2 + 1) = 0$$
$$2x_1 - 3x_2 - 5 = 0.$$

c) The first and second forward-jump operators of time scale $\Lambda_3 = \mathbb{Z} \times \mathbb{R}$ becomes $\sigma_1(t) = t + 1$ and $\sigma_2(t) = t$, respectively. In Figure3, the delta gradient of the level surface of Λ_3 is obtained as follows, see Figure 3.

$$\nabla_\Delta f = (2[\sigma_1(x_1) + x_1], -3[\sigma_2(x_2) + x_2])$$
$$= (2[x_1 + 1 + x_1], -3[x_2 + x_2])$$
$$= (4x_1 + 2, -6x_2).$$

If $\nabla_\Delta f = (4x_1 + 2, -6x_2)|_{P(1,-1)} = (6,6) \neq 0$, then this point is a regular point on the level surface. The normal of the gradient vector is

$$\| \nabla_\Delta f \| = \sqrt{6^2 + 6^2} = \sqrt{72} = 6\sqrt{2}.$$

The delta derivative in the direction of the unit vector $\vec{u} = (1, -1)$ becomes

$$\frac{\partial f}{\Delta \vec{u}} = \nabla_\Delta f . \vec{u}$$
$$= < (4x_1 + 2, -6x_2), (1, -1) >$$
$$= 4x_1 + 6x_2 + 2.$$

Thus, the maximum value of $\partial f/\Delta \vec{u}$ at point $P(1,-1)$ is $\| \nabla_\Delta f(P) \| = 6\sqrt{2}$. Next, the unit normal vector of the level surface at point $P(1,-1)$ is obtained using the following equation

$$\vec{n} = \frac{\nabla_\Delta f}{\| \nabla_\Delta f \|} = \frac{(6,6)}{6\sqrt{2}} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}).$$

Finally, the tangential plane of the level surface at point $P(1,-1)$ is given by

$$(\vec{R} - \vec{R}_0).\nabla_\Delta f(P_0) = 0$$
$$[(x_1, x_2) - (1,-1)].(6,6) = 0$$
$$(x_1 - 1, x_2 + 1).(6,6) = 0$$
$$(x_1 - 1, x_2 + 1).(1,1) = 0$$
$$x_1 - 1 + x_2 + 1 = 0$$
$$x_1 + x_2 = 0$$

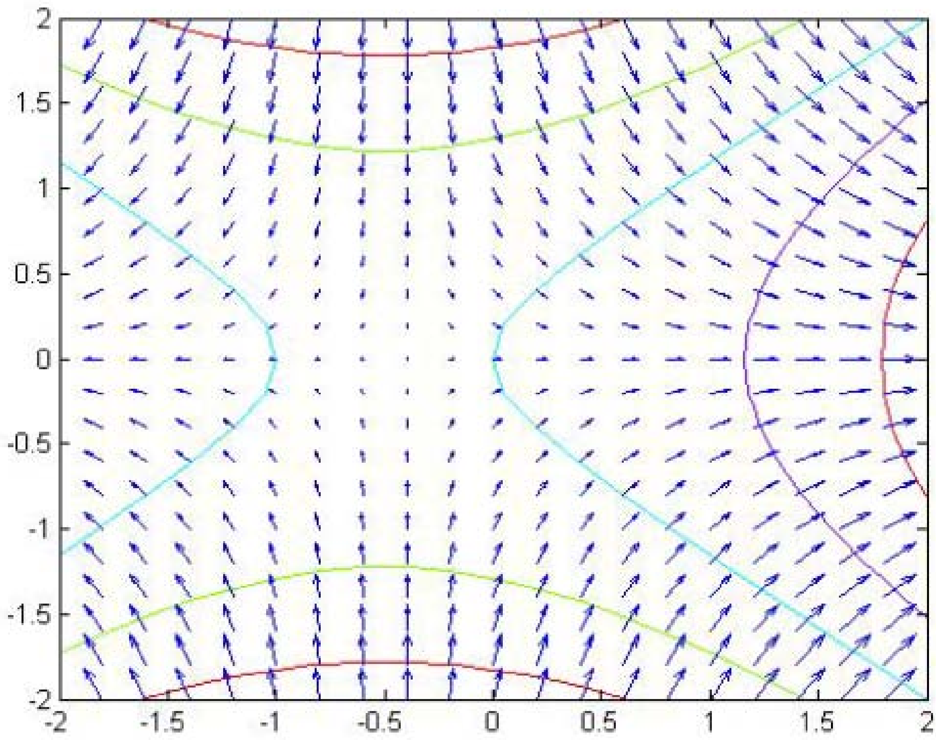

Fig. 3 The gradient vector field $\nabla_\Delta f$ on $\Lambda_3 = \mathbb{Z} \times \mathbb{R}$.

4. Conclusion

In this paper it is seen that the level surface and the delta gradient function, which are frequently used in geometric analyses and physical scenarios involving continuous derivatives, can be solved using both continuous and discrete spaces that are simultaneously obtained in time. As such, the geometry of time scales plays an important role in unifying discrete and continuous geometry, making it possible to use the properties of the level surface and the delta gradient functions in many fields of discrete and continuous geometry.

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