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The Apprentices' Tower of Hanoi

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Abstract: In this paper, we consider a variant on the classic Tower of Hanoi puzzle. Unlike the original puzzle, our variation allows the player to place a larger disc directly on top of a smaller one provided that all other discs on the peg are in increasing order of their diameter. We call this variation the *Apprentices' Tower of Hanoi*. We give an upper bound on the number of moves to transfer all the discs from one peg to another in this variation. We also give several graph theoretic properties of the associated Hanoi graph.

Key words: Tower of Hanoi

1. Introduction

The Tower of Hanoi was introduced in 1883 by Édouard Lucas. The traditional puzzle consists of three pegs and n discs, each of a different diameter. The puzzle starts with all discs on the first peg arranged in such a way that the diameter of the discs increases from top to bottom. The goal of the puzzle is to transfer all discs to the third peg under the following stipulations:

i) Only one disc may be moved at a time;

ii) Only the top disc on each peg can be moved;

iii) The Divine Rule—A larger disc can never be placed on top of a smaller one.

As shown by Wood [1], it takes a minimum of $2^n - 1$ moves to accomplish this task. Lucas describes a legend in which sixty-four discs were placed on the first peg at the beginning of time. A group of monks work tirelessly transferring discs at a rate of one per second. Lucas describes that when the monks have completed their task, the world will end. Even if this legend were true, we would have little to worry about as it would take the monks five billion centuries to complete their task [2].

As shown in the 2013 text by Hinz et al. [3], there are many variations on the Tower of Hanoi puzzle. One variation involves increasing the number of available pegs as in the Reve's Puzzle [4] or the infamous Frame-Stewart Conjecture [5]. Many variations relax the Divine Rule. These variations include the Bottleneck Tower of Hanoi [6], the Santa Claus Tower, and the Sinners' Tower [7]. In this paper, we consider another variation in which the Divine Rule has been relaxed. We modify Lucas's legend to justify this variation.

The monks who work at the Tower of Hanoi have been doing so since the dawn of time. As such they have become quite old and tired. It is with this in mind that they have decided to train a new generation of monks to take over the transfer of the discs. If left to their own accord, then the young apprentices might ignore the rules for the transfer process. However, they are under the watchful eyes of the vigilant monks whose sworn duty is to ensure that the rules are obeyed. The wise old monks can keep track of how many golden discs are moved at once, and guarantee that only the top golden disc on any peg is moved. Unfortunately, with age comes poor eye-sight. Thus, so long as the stacks only have one misplaced golden disc per peg, the senior monks will not notice. Does this spell doom for us all?

We call this variation *the Apprentices' Tower of Hanoi*. In this variation, a larger disc can be placed directly on top of another provided that the remaining discs on the peg are in *regular order*, that is, in

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increasing order of their diameter from top to bottom. In other words, the player is allowed to have at most one "sin" at any given time on each peg. The goal of this paper is to gain insight into this variation of the Tower of Hanoi.

2. The Hanoi Graph

Hanoi graphs are a common method of exploring a Tower of Hanoi type puzzle. In a *Hanoi graph* every possible state of the puzzle is represented by a vertex. Two vertices are adjacent in the Hanoi graph if their corresponding states differ by one move. In this section, we examine several properties of the Hanoi graph for the Apprentices' Tower. Let AH_n denote the Apprentices' Tower of Hanoi graph on n discs. For all undefined graph theory terminology, refer to Chartrand [8].

Proposition 2.1. The graph AH_n is connected for all n.

Proof. As shown in [3] any irregular state in the traditional Tower of Hanoi can be transformed into a regular one using moves that obey the Divine Rule. In particular, this holds for the states in the Apprentices' Tower in which there is at most one sin per peg. Thus, the corresponding graph AH_n is connected.

Our notation for states in the Apprentices' Tower is consistent with the notation used in [3] for states in which the discs do not obey the Divine Rule. Namely, we represent each state as

$$x_1 \dots x_{\lambda_1} | y_1 \dots y_{\lambda_2} | z_1 \dots z_{\lambda_3}.$$

This means that on peg 1, disc x_1 is on top of disc x_2, which is in turn on top of x_3, and so on. Similarly, y_1 is the top disc on peg 2, followed by $y_2, \dots, y_{\lambda_2}$. Analogously, the discs of peg 3 from top to bottom are $z_1, \dots, z_{\lambda_3}$. We can think of each of these arrangements as a permutation on λ_i distinct symbols. Note that a permutation $\pi_1 \dots \pi_n$ has a *fall* if $\pi_i > \pi_{i+1}$ for any i. Since an apprentice can place a larger disc directly on top of a smaller one once per peg, this means that there is at most one fall in the permutation. With this in mind, we give the number of permutations with at most one fall.

Proposition 2.2. The number of permutations on λ with at most one fall is given by $2^\lambda - \lambda$ [9].

Using this, we can compute the number of states in the Apprentices' Tower of Hanoi. Equivalently, this will give the number of vertices in the Hanoi graph.

Theorem 2.3. The number of vertices in the Apprentices' Tower of Hanoi graph is given by

$$6^n - 3n5^{n-1} + 3n(n-1)4^{n-2}$$
$$-n(n-1)(n-2)3^{n-3}.$$

Proof. We count this in $\frac{(n+1)(n+2)}{2}$ disjoint, exhaustive sets. In this case, the $(\lambda_1, \lambda_2, \lambda_3)$th set is the class of all configurations in which λ_i discs are placed on the ith peg. This set can be counted by:

i) Selecting λ_i discs to place on disc i for $i = 1,2,3$. The number of ways to do this is given by the multinomial coefficient, $\binom{n}{\lambda_1, \lambda_2, \lambda_3}$;

ii) Arranging the λ_1 discs on peg 1 in such a way that there is at most one fall. There are $2^{\lambda_1} - \lambda_1$ ways to do this by Proposition 2.2;

iii) Arranging the λ_2 discs on peg 2 in such a way that there is at most one fall. There are $2^{\lambda_2} - \lambda_2$ ways to do this by Proposition 2.2;

iv) Arranging the λ_3 discs on peg 3 in such a way that there is at most one fall. There are $2^{\lambda_3} - \lambda_3$ ways to do this by Proposition 2.2.

By the Multiplication Principle, the cardinality of the $(\lambda_1, \lambda_2, \lambda_3)$th set is

$$\binom{n}{\lambda_1, \lambda_2, \lambda_3} (2^{\lambda_1} - \lambda_1)(2^{\lambda_2} - \lambda_2)(2^{\lambda_3} - \lambda_3).$$

Summing up over all values of λ_i such that $\lambda_1 + \lambda_2 + \lambda_3 = n$ and $\lambda_i \geq 0$ yields

$$\sum_{\substack{\lambda_1+\lambda_2+\lambda_3=n \\ \lambda_i \geq 0}} \binom{n}{\lambda_1, \lambda_2, \lambda_3} (2^{\lambda_1} - \lambda_1)(2^{\lambda_2} - \lambda_2)(2^{\lambda_3} - \lambda_3).$$

To obtain the expression given, we expand the above equation. The terms in this expression are the multinomial coefficient $\binom{n}{\lambda_1, \lambda_2, \lambda_3}$ multiplied by 2^n, $\lambda_i 2^{\lambda_j + \lambda_k}$, $\lambda_i \lambda_j 2^{\lambda_k}$, and $\lambda_1 \lambda_2 \lambda_3$. These terms can be simplified by using the following identities derived from the Multinomial Theorem:

$$6^n = \sum_{\substack{\lambda_1+\lambda_2+\lambda_3=n \\ \lambda_i \geq 0}} \binom{n}{\lambda_1, \lambda_2, \lambda_3} 2^{\lambda_1} 2^{\lambda_2} 2^{\lambda_3},$$

$$n5^{n-1} = \sum_{\substack{\lambda_1+\lambda_2+\lambda_3=n \\ \lambda_i \geq 0}} \binom{n}{\lambda_1, \lambda_2, \lambda_3} \lambda_1 2^{\lambda_2} 2^{\lambda_3},$$

$$n(n-1)4^{n-2} = \sum_{\substack{\lambda_1+\lambda_2+\lambda_3=n \\ \lambda_i \geq 0}} \binom{n}{\lambda_1, \lambda_2, \lambda_3} \lambda_1 \lambda_2 2^{\lambda_3}, \text{ and}$$

$$n(n-1)(n-2)3^{n-3} = \sum_{\substack{\lambda_1+\lambda_2+\lambda_3=n \\ \lambda_i \geq 0}} \binom{n}{\lambda_1, \lambda_2, \lambda_3} \lambda_1 \lambda_2 \lambda_3.$$

Replacing these expressions into the above expansion yields the desired result.

Note that the traditional Tower of Hanoi on n discs has 3^n states [3]. Theorem 2.3 shows that the number of states in the Apprentices' Tower is asymptotic to 6^n. This suggests that determining the minimum number ofmoves necessary for solving the Apprentices' Tower is a more difficult problem than the traditional Tower of Hanoi.

We now consider the planarity of the Apprentices' Tower of Hanoi graph. Note that the traditional Tower of Hanoi graph is planar for all values of n [3]. In our next result, we determine necessary and sufficient conditions for the Apprentices' Tower of Hanoi graph to be planar.

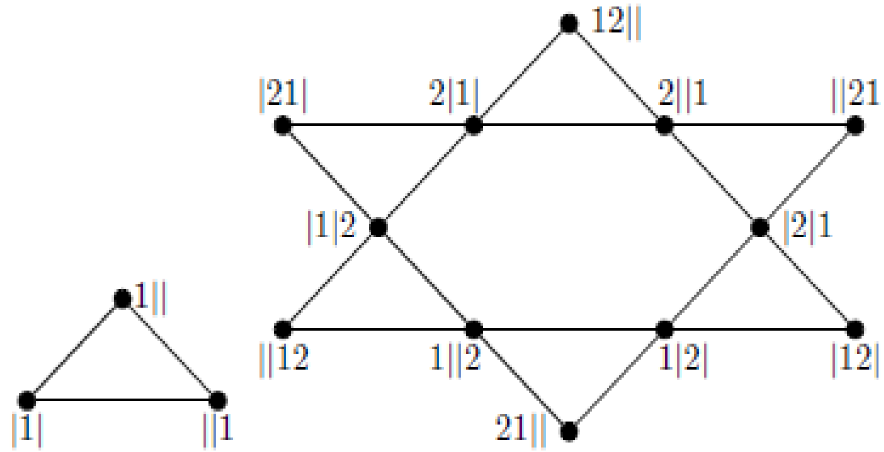

Fig. 1 Apprentices' Tower of Hanoi graphs for $\mathbf{n = 1, 2}$.

Theorem 2.4. The Apprentices Tower of Hanoi graph on n discs is planar if and only if $n \leq 2$.

Proof. The Apprentices' Tower of Hanoi graphs on one and two discs are clearly planar, as shown in Figure 1. Since AH_n is a subgraph of AH_k for all $k \geq n$, it suffices to show that AH_3 is non-planar. By Kuratowski's Theorem [8], to show that AH_3 is non-planar it suffices to show that AH_3 has a subdivision of the complete bipartite graph $K_{3,3}$. Define $x_1 = 3|1|2$, $x_2 = 1|2|3$, $x_3 = 2|3|1$, $y_1 = 1|3|2$, $y_2 = 3|2|1$,and $y_3 = 2|1|3$. The paths from the x_i to the y_j are given below:

$$x_1 \mapsto 3||12 \mapsto |3|12 \mapsto y_1; x_1 \mapsto 23|1| \mapsto 23||1 \mapsto y_2;$$

$$x_1 \mapsto 3|21| \mapsto |21|3 \mapsto y_3; x_2 \mapsto 31|2| \mapsto 31||2 \mapsto y_1;$$

$$x_2 \mapsto |12|3 \mapsto 3|12| \mapsto y_2; x_2 \mapsto 1||23 \mapsto ||123 \mapsto |1|23 \mapsto y_3;$$

$$x_3 \mapsto |23|1 \mapsto 1|23| \mapsto y_1; x_3 \mapsto |3|21 \mapsto 3||21 \mapsto y_2;$$

$$x_3 \mapsto 32||1 \mapsto 32|1| \mapsto y_3.$$

In the traditional Hanoi graph, the maximum degree is either two (in the case with one disc) or three (for all other values of n). The minimum degree on the traditional Hanoi graph is two for all numbers of discs. We now specify the maximum degree and the minimum degree for AH_n.

Theorem 2.5. The minimum degree for AH_n is two for all $n \geq 1$. The maximum degree for AH_1 is two. The maximum degree for AH_2 is four. For $n \geq 3$, the maximum degree of AH_n is six.

Proof. Since $n \geq 1$, there is at least one peg with at least one disc. Among all non-empty pegs, choose the one in which the top disc is of minimum diameter. This disc can move to either peg, thus the minimum degree is at least two. The minimum degree of two is achieved by taking any state in which all of the discs are on a single peg.

As shown in Figure 1, the maximum degree for AH_1 is two. Likewise, Figure 1 illustrates that the maximum degree of AH_2 is four. Because only the top disc can move and the top disc has at most two possible destinations, the maximum degree for AH_n is at most six. In order for each of the three top discs to have two possible destinations, clearly no peg can be empty. Further, if the Divine Rule holds on each peg, then we guarantee that each to the top discs can move to either of the remaining pegs. The smallest n for which this is achieved is $n = 3$. For example, $1|2|3$ is adjacent to $|12|3$, $|2|13$, $21||3$, $1||23$, $31|2|$, and $1|32|$. Thus, the maximum degree of AH_n is six for $n \geq 3$.

In our final result in this section, we find the chromatic number for AH_n.

Theorem 2.6. For all n, the chromatic number for AH_n is three.

Proof. Let H_n denote the traditional Hanoi graph on n discs. By [3], the chromatic number of H_n is three for all n. Note that this is a subgraph of AH_n. Hence, the chromatic number of AH_n is at least three. Color the H_n subgraph using a proper 3-coloring. Suppose pegs 1, 2, and 3 have m, ℓ, and q discs, respectively. Any other configuration with m, ℓ, and q discs on pegs 1, 2, and 3, respectively, will not be adjacent to this state or to each other. Hence, they may be placed into the same color class. Since one of these configurations has all pegs in a regular order, it follows AH_n has the same number of color classes as H_n.

3. An Algorithm and an Upper Bound

In this section, we give an upper bound on the minimum number of moves necessary for solving the Apprentices' Tower of Hanoi. To facilitate this, we define $S_n(s_1, s_2, s_3)$ to be the minimum number of moves necessary to transfer n discs (in increasing order of diameter from top to bottom) from peg 1 to peg 3 (in increasing order of diameter from top to bottom) such that the player is allowed to break the Divine Rule at most s_i times on peg i. Note that $S_n(0,0,0)$ is the minimum number of moves necessary to solve the traditional Tower of Hanoi puzzle. Further, $S_n(1,1,1)$ is the minimum number of moves necessary to solve the Apprentices' Tower of Hanoi puzzle defined above.

Proposition 3.1. For all n, s_1, s_2, and s_3, $S_n(s_1, s_2, s_3) = S_n(s_3, s_2, s_1)$.

Proof. By definition, $S_n(s_1, s_2, s_3)$ is the minimum number of moves required to move a regular stack of n discs to another peg in regular order. So, reversing this sequence of moves will be the minimum number required to return the discs to the first peg.

Proposition 3.2. For all n, s_1, s_2, and s_3, we have that

$$2n - 1 \leq S_n(s_1, s_2, s_3) \leq 2^n - 1.$$

Further, these bounds are sharp.

Proof. It takes a minimum of $2^n - 1$ moves to solve the traditional Tower of Hanoi (i.e., $S_n(0,0,0) = 2^n - 1$). Thus, we can solve the variant using only moves allowed on the traditional tower. Hence, the upper bound follows and it is sharp

In order to move the largest disc, peg 1 must contain only the largest disc. It takes at least $n - 1$ moves to transfer the top $n-1$ discs from the first peg to the second. We then move the largest disc to the third peg. We then move the remaining $n-1$ discs from the second peg to the third. This requires at least $n-1$ moves. Hence, $S_n \geq n-1+1+n-1 = 2n-1$.

Clearly, the lower bound is sharp for $n=2$ or for $s_2 \geq n-2$.

The next theorem not only provides a bound, but an efficient algorithm. Our strategy in Theorem 3.3 is similar to the "tower-splitting" strategy used in the Frame-Stewart Conjecture.

Theorem 3.3. For all n, s_1, s_2, s_3, and k such that $s_2 \geq 1$ and $1 \leq k \leq n$:

$$S_n(s_1,s_2,s_3) \leq$$
$$S_{n-k}(s_1,s_3,s_2) + S_{n-k}(s_2,s_1,s_3) +$$
$$S_{k-1}(s_1,s_3,s_2-1) + S_{k-1}(s_2-1,s_1,s_3) + 1.$$

Proof. It is sufficient to give an algorithm that solves the puzzle in the required number of moves for all k.

i) Move the top $\mathrm{n-k}$ discs to the second peg in a regular order. This takes $\mathrm{S_{n-k}(s_1,s_3,s_2)}$ moves by definition;

ii) Move the remaining $\mathrm{k-1}$ discs to the second peg in a regular order. We now have one less "sin" to work with on this peg. Hence, this takes $\mathrm{S_{k-1}(s_1,s_3,s_2-1)}$ moves;

iii) Move the largest disc to the third peg. This takes one move;

iv) Move the top $\mathrm{k-1}$ discs from the second peg to the third peg. This takes $\mathrm{S_{k-1}(s_2-1,s_1,s_3)}$ moves;

v) Move the remaining $\mathrm{n-k}$ discs from the second peg to the third peg. This takes $\mathrm{S_{n-k}(s_2,s_1,s_3)}$ moves.

The total number of moves used in the above algorithm is

$$S_{n-k}(s_1,s_3,s_2) + S_{n-k}(s_2,s_1,s_3) +$$
$$S_{k-1}(s_1,s_3,s_2-1) + S_{k-1}(s_2-1,s_1,s_3) + 1.$$

Since this holds for all k, the result holds.

Note that in the Apprentices' Tower of Hanoi, $s_1 = s_2 = s_3 = 1$. Hence, Theorem 3.3 yields

$$S_n(1,1,1) \leq \min_k\{\, 2S_{n-k}(1,1,1) + S_{k-1}(1,1,0)$$
$$+ S_{k-1}(0,1,1) + 1\}.$$

Table 1 The conjectured values of $\mathbf{S_n(1,1,1)}$ for small n.

n	1	2	3	4	5	6	7	8	9	10
$f(n)$	1	3	5	9	13	17	25	29	37	45

Applying Proposition 3.1 yields

$$S_n(1,1,1) \leq \min_k\{\, 2S_{n-k}(1,1,1) + 2S_{k-1}(1,1,0) + 1\}.$$

In Table 1, we provide values for the function that $f(n) = 2S_{n-k}(1,1,1) + 2S_{k-1}(1,1,0) + 1$, where $k = \lfloor n/2 \rfloor$. As shown in the table, this function grows very slowly. So if the legend were true and the apprentices were allowed to do as they please, then the world would be doomed.

To show that the algorithm in Theorem 3.3 is optimal, we would need to show that among all optimal algorithms, there is one that uses the strategy described above. In other words:

i) The largest disc moves at most once in an optimal algorithm;

ii) Splitting the tower by moving the top $\mathrm{n-k}$ discs to the second peg is necessary in an optimal algorithm.

If both of the above assumptions were true, then it would give a proof of the following conjecture:

Conjecture 3.4. For all n,

$$S_n(1,1,1) = \min_k\{\, 2S_{n-k}(1,1,1) + 2S_{k-1}(1,1,0) + 1\}.$$

However, it may be quite difficult to prove that those two assumptions are necessary in an optimal algorithm. As the "tower splitting" assumption is also an unproven assumption in the Frame-Stewart Conjecture, Conjecture 3.4 may remain unproven for some time.

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