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The Problem of Multi-soliton Collision for Essentially Nonintegrable Equations

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Abstract: We consider a nonuniqueness phenomenon which appears in the framework of the weak description of multi-soliton collision for nonintegrable equations. The general idea is realized in the case of two waves and for the KdV-type equation with small dispersion and nonlinearity u^4.

Key words: generalized Korteweg-de Vries equation, soliton, interaction, weak asymptotics method, weak solution, non-integrability

1. Introduction

1.1. Statement of the Problem

The main question under consideration in this paper is the following: do the integrable equations form a compact cluster with a sharp frontier or there are nonintegrable equations which preserve in a sense some properties of integrability? We guess this problem can't be solved nowadays in the general setting. Therefore we restrict ourself to the generalized Korteweg-de Vries equations, specifically to the gKdV-4 equation,

$$\frac{\partial u}{\partial t}+\frac{\partial u^{\mu}}{\partial x}+\varepsilon^2\frac{\partial^3 u}{\partial x^3}=0,\quad x\in\mathbb{R}^1,\qquad t>0,\quad \varepsilon<<1, \tag{1}$$

$\mu=4$, and we consider the scenario of solitary wave collision only.

It is well known that an arbitrary number of solitary waves collide for integrable nonlinear equations in an remarkable manner: they pass through each other almost as linear waves. In particular, it is true for (1) with $\mu=2$ and $\mu=3$. Moreover, there exist explicit N-phase formulas which describe this collision. Conversely, for μ other than two or three there are neither explicitly the same manner of interaction nor explicit N-phase formulas. However, its possible to prove that in an asymptotic sense two waves interact almost elastically for each $\mu\in(1,5)$ [1-2]. More in detail, let us consider the linear combination of two perturbed solitary waves

$$u=\sum_{i=1}^{2}G_i\,\omega\left(\beta_i\frac{x-\varphi_i}{\varepsilon}\right) \tag{2}$$

with variable amplitudes $G_i=G_i(t,\varepsilon)$ and nonlinear phases $\varphi_i=\varphi_i(t,\varepsilon)$. Here ω is a function such that $A\omega(\beta(x-\varphi_0)/\varepsilon)$ with $A=\text{const}$ and

$$\beta=\sqrt{\frac{2}{\mu+1}}A^{\frac{\mu-1}{2}},\qquad \varphi_0=Vt+\text{const},\quad V=\beta^2 \tag{3}$$

represents the explicit solitary wave solution of (1) with the normalization condition $\omega(0)=1$. We assume also that $\varphi_1(0,\varepsilon)-\varphi_2(0,\varepsilon)=\text{const}>0$ and $G_2|_{t=0}=A_2>G_1|_{t=0}=A_1$. Thus, the function (2) at $t=0$ represents two solitons with the amplitudes A_i which are concentrated near the points $\varphi_i(0,\varepsilon)$, whereas u has the value $O(\varepsilon^\infty)$ for $x\in(\varphi_2(0,\varepsilon)+c\varepsilon^{1-\gamma},\varphi_1(0,\varepsilon)-c\varepsilon^{1-\gamma})$, $\gamma>0$, and $\varepsilon\to 0$. Moreover, the trajectories $x=V_it+\varphi_i(0,\varepsilon)$, $i=1,2$, intersect at a point (x^*,t^*).

Furthermore, let us introduce solutions in the weak asymptotic sense:

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Definition 1. A sequence $u(t,x,\varepsilon)$, belonging to $\mathcal{C}^\infty(0,T;\mathcal{C}^\infty(\mathbb{R}_x^1))$ for $\varepsilon = \text{const} > 0$ and belonging to $\mathcal{C}(0,T;\mathcal{D}'(\mathbb{R}_x^1))$ uniformly in $\varepsilon \geq 0$, is called a weak asymptotic mod $O_{\mathcal{D}'}(\varepsilon^2)$ solution of (1) if the relations

$$\frac{d}{dt}\int_{-\infty}^{\infty} u\psi dx - \int_{-\infty}^{\infty} u^\mu \frac{\partial \psi}{\partial x} dx = O(\varepsilon^2), \tag{4}$$

$$\frac{d}{dt}\int_{-\infty}^{\infty} u^2 \psi dx - \int_{-\infty}^{\infty}\left\{\frac{2\mu}{\mu+1}u^{\mu+1} - 3\left(\varepsilon\frac{\partial u}{\partial x}\right)^2\right\}\frac{\partial \psi}{\partial x}dx = O(\varepsilon^2) \tag{5}$$

hold uniformly in t for any test function $\psi = \psi(x) \in \mathcal{D}(\mathbb{R}^1)$.

Here the right-hand sides are $\mathcal{C}^\infty$-functions for $\varepsilon = \text{const} > 0$ and piecewise continuous functions uniformly in $\varepsilon \geq 0$. The estimates are understood in the $\mathcal{C}(0,T)$ sense:

$$g(t,\varepsilon) = O(\varepsilon^k) \leftrightarrow \max_{t\in[0,T]} |g(t,\varepsilon)| \leq c\varepsilon^k.$$

The main previous result is the following [1-2]:

Theorem 1. Let $\theta^* = \theta^*(\mu)$ be a real positive solution of the equation

$$1 - \theta^* = 2(\theta^*)^\kappa, \quad \kappa = 2/(\mu - 1), \tag{6}$$

and let $\theta \stackrel{\text{def}}{=} \beta_1/\beta_2 < \theta^*$. Then there exist functions $G_i(t,\varepsilon)$, $\varphi_i = \varphi_i(t,\varepsilon)$, $i = 1,2$, such that (2) describes mod $O_{\mathcal{D}'}(\varepsilon^2)$ the elastic scenario of the solitary waves interaction.

The next theorem allows to treat the weak asymptotics (2) in the classical sense [1-2]:

Theorem 2. Let $\theta < \theta^*$. Then the function u of the form (2) is a weak asymptotic mod $O_{\mathcal{D}'}(\varepsilon^2)$ solution of (1) if and only if u satisfies the following conservation and balance laws:

$$\frac{d}{dt}\int_{-\infty}^{\infty} u\, dx = 0, \quad \frac{d}{dt}\int_{-\infty}^{\infty} u^2\, dx = 0, \tag{7}$$

$$\frac{d}{dt}\int_{-\infty}^{\infty} x\, u^2 dx - \int_{-\infty}^{\infty} u^\mu\, dx = 0, \tag{8}$$

$$\frac{d}{dt}\int_{-\infty}^{\infty} x\, u^2 dx - 2\frac{\mu}{\mu+1}\int_{-\infty}^{\infty} u^{\mu+1}\, dx + 3\int_{-\infty}^{\infty}\left(\varepsilon\frac{\partial u}{\partial x}\right)^2 dx = 0. \tag{9}$$

Thus, in the weak asymptotic sense we obtain again both explicit formulas and elastic type of interaction. Note that we consider the waves of arbitrary amplitudes but assume the smallness of the dispersion ε. The last is equivalent to the consideration of large distances and time interval. Moreover, it turned out that the solitons collide elastically (in the leading term) in the case when the condition $\theta < \theta^*$ is violated, see [3] for the case $\mu = 4$, $\theta = 0.56$, and [2] for the case $\mu = 3/2$, $\theta = 0.84$ respectively, whereas $\theta^*(4) = \sqrt{5} - 2 \sim 0.24$ and $\theta^*(3/2) \sim 0.457$. Another example is depicted in Fig.1 for $\mu = 4$ and $\theta = 0.95$. Note also that the oscillating tail which appears after the collision is not a computational error but a part of the solution [4]. Obviously we can treat it as a "radiation" which is well-known for the perturbed KdV equation.

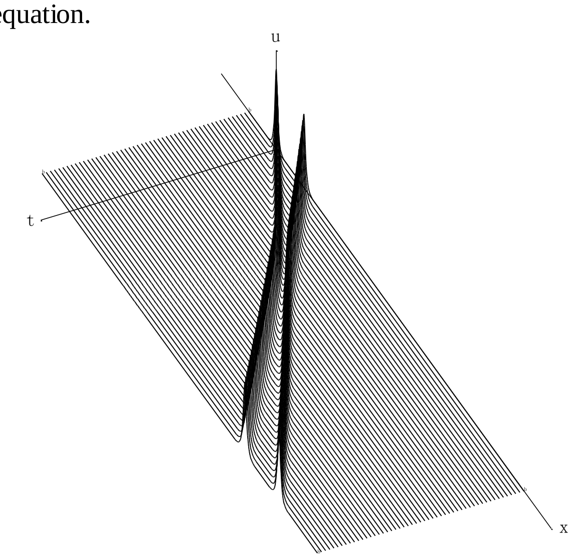

Fig. 1 Evolution of two solitary waves for $\beta_1 = 1.75$, $\beta_2 = 2$, and $\varepsilon = 0.1$.

Unfortunately, the conclusion about the almost elastic scenario of interaction for two waves only doesn't imply that (1) preserves asymptotically properties of integrability. The point is such that there are known equations fundamentally distinct from the integrable ones, but with explicit 2-phase formulas [5]. That is why we should consider at least three waves to declare that (1) is close to KdV and MKdV in an asymptotic sense. It turned out however that Definition 1 doesn't support asymptotics with three or more phases since it implies the appearance of ill-posed model equations for the parameters of the solutions (they are well-posed for the case of two phases). To overcome the obstacle it is necessary to change the viewpoint on the weak asymptotic solution: the analysis in [6] showed that for two-phase solutions the definition 1 implies the fulfilment of two conservation laws in the weak sense. Moreover, the one-phase asymptotic theory for perturbed equations implies the fulfilment of a single "conservation law" again in the weak sense. Thus there appears the hypotheses that to construct N-phase asymptotics it is necessary to use N conservation laws. Appropriate construction in [6] verified this idea and more detailed analysis [7] showed that three waves collide elastically again (see also Fig 2). At the same time these results induce other questions: what to do when the quantity of interacting waves is greater than the number of existing conservation laws; and conversely when the quantity of interacting waves is less than the number of conservation laws, how to select conservation laws between all existing?

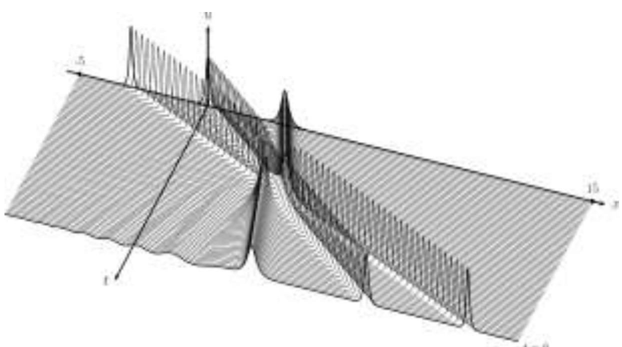

Fig. 2 Evolution of the soliton triplet with $\beta_1 = 0.7$, $\beta_2 = 1.25$, $\beta_3 = 1.62$, and $\varepsilon = 0.1$.

The aim of this paper is to answer these questions. The main result is very simple: instead of deficient conservation laws it is possible to use balance laws related with reasonable *a-priory* estimates, whereas the choice of specific conservation or balance laws is not important. Let us note that the situation with nonuniqueness here is inverse to the case of shock waves for hyperbolic equations: for shocks there are many divergent forms for the original classical equation but we should fix only one from them; conversely, for solitons we should use both all divergent forms and balance laws.

1.2 Weak Asymptotics Method: The Main Idea

The basic remark is very simple: rapidly varying solitary wave solutions (soliton or kink type) tend to distributions as the small parameter tends to zero. This allows to treat the equation in the weak sense and, respectively, look for singularities instead of regular functions. Obviously, non-integrability implies that we cannot find neither classical nor weak exact solutions. However, we can construct an asymptotic weak solution considering the smallness of the remainder in the weak sense. The main advantage here is such that to construct asymptotics it is enough to obtain and analyze ordinary differential equations instead of partial differential equations. In a sense, the situation here is similar to the shock waves: various regularization generates various profile for the wave, but in the limiting passage we obtain the same Rankine-Hugoniot conditions. For solitons the passage to the weak expansion results in the disappearance of the shape, but preserves the soliton's characteristics: amplitudes and phases. For the problem of interaction these parameters vary in a neighborhood of the time instant of the collision and stabilize ourself after that. Deriving uniform in time model equations for the parameters we can describe the scenario of the wave interaction.

Originally, such idea had been suggested by V. Danilov and V. Shelkovich for shock wave type solutions (1997, [8]), then generalized for soliton type solutions (V. Danilov and G. Omel'yanov 2003 [1]), and it has been developed and adapted later for many other problems (V. Danilov, G. Omel'yanov, V. Shelkovich, D. Mitrovic and others, [2,6,7], [9-15] and references therein). Let us note finally that the treatment [6] of weak asymptotics as functions which satisfy some conservation laws or energy relations takes us back to the ancient Whitham's idea to construct one-phase asymptotic solution satisfying a Lagrangian. Now, for essentially nonintegrable equations and multi-phase solutions, we use the appropriate number of the laws and satisfy them in the weak sense.

2. Asymptotics Construction

To simplify formulas we consider here and in what follows the GKdV-4 equation only. Respectively, we obtain for A, β, and the function ω from (2), (3):

$$\omega(\eta) = \{\cosh(\eta/\gamma)\}^{-\gamma}, \quad A = c\beta^{\gamma}, \quad c = (5/2)^{1/3}, \quad \gamma = 2/3. \tag{10}$$

The GKdV-4 equation has three conservation laws, which we write in the differential form with the remainder:

$$\frac{\partial Q_i}{\partial t} + \frac{\partial P_i}{\partial x} = O_{\mathcal{D}'}(\varepsilon^2), \quad i = 1,2,3. \tag{11}$$

Here

$$Q_1 = u, \quad P_1 = u^4, \quad Q_2 = u^2, \quad P_2 = \frac{8}{5}u^5 - 3(\varepsilon u_x)^2, \tag{12}$$

$$Q_3 = (\varepsilon u_x)^2 - \frac{2}{5}u^5, \quad P_3 = 16u^3(\varepsilon u_x)^2 - u^8 - 3(\varepsilon^2 u_{xx})^2, \tag{13}$$

and we use the following definition of the smallness:

Definition 2. A function $v(t,x,\varepsilon)$ is said to be of the value $O_{\mathcal{D}'}(\varepsilon^k)$ if the relation

$$\int_{-\infty}^{\infty} v(t,x,\varepsilon)\psi(x)dx = O(\varepsilon^k)$$

holds uniformly in t for any test function $\psi \in \mathcal{D}(\mathbb{R}_x^1)$.

The first conservation law (11) is the equation (1) itself in the sense of Definition 1, whereas the others are the result of multiplication of (1) by u and $u^4 + \varepsilon^2 u_{xx}$ respectively. Furthermore, let us multiply (1) by

$$\varepsilon^2 \frac{\partial^2}{\partial x^2}(u^4 + \varepsilon^2 u_{xx}) + \frac{8}{3}\varepsilon^2 u^3 u_{xx} - 2\varepsilon^2 u^2 u_x^2 + \frac{40}{21}u^7.$$

Then we obtain the balance law

$$\frac{\partial Q_4}{\partial t} + \frac{\partial P_4}{\partial x} + \varepsilon^{-1}K_4 = O_{D'}(\varepsilon^2), \tag{14}$$

where

$$Q_4 = \frac{1}{2}(\varepsilon^2 u_{xx})^2 + \frac{5}{21}u^8 - \frac{10}{3}u^3(\varepsilon u_x)^2, \quad K_4 = -(\varepsilon u_x)^5, \tag{15}$$

$$P_4 = 12u^3(\varepsilon^2 u_{xx})^2 - 19u(\varepsilon u_x)^4 - \frac{3}{2}(\varepsilon^3 u_{xxx})^2 + \frac{160}{231}u^{11} - \frac{100}{3}u^6(\varepsilon u_x)^2.$$

We will write (11) in the form (14) setting $K_j = 0$ for $j = 1,2,3$. Let us note that, in contrast to Q_j and P_j, the non-divergent term $\varepsilon^{-1}K_4$ ("production") doesn't belong to the so called "regularly degenerating" functions, so that its value is $O(\varepsilon^{-1})$ in the C-sense, that is the same as the value of the derivatives of Q_j and P_j. At the same time, $\varepsilon^{-1}K_4$ calculated for single solitary wave $A\omega(\beta(x-\varphi_0)/\varepsilon)$ is an odd function and it disappears after the integration.

Let us consider again two-phase asymptotic solution for the GKdV-4 equation supplied by the initial condition

$$u|_{t=0} = \sum_{i=1}^{2} A_i\, \omega\left(\beta_i \frac{x - x_{i0}}{\varepsilon}\right). \tag{16}$$

Contrarily to Definition 1 we define it in the following manner:

Definition 3. Let $1 \le k_0 < k_1 \le 4$ and let a sequence $u_{k_0,k_1} = u_{k_0,k_1}(t,x,\varepsilon)$ belong to the same functional space as $u(t,x,\varepsilon)$ in Definition 1. Then u_{k_0,k_1} is called a weak asymptotic mod $O_{\mathcal{D}'}(\varepsilon^2)$ solution of (1) if the relations

$$\frac{\partial Q_j}{\partial t} + \frac{\partial P_j}{\partial x} + \varepsilon^{-1}K_j = O_{\mathcal{D}'}(\varepsilon^2), \quad j = k_0, k_1 \tag{17}$$

hold uniformly in t.

To construct the asymptotics we present the ansatz u_{k_0,k_1} again in the form (2), where

$$G_i = A_i + S_i(\tau),\ \varphi_i = \varphi_{i0}(t) + \varepsilon\varphi_{i1}(\tau),\ \tau = \beta_1(\varphi_{20}(t) - \varphi_{10}(t))/\varepsilon, \tag{18}$$

A_i are the original amplitudes and $\varphi_{i0} = V_i t + x_{i0}$ describe the trajectories of the noninteracting waves; β_i, A_i, and V_i are connected by the equalities (3), (10); the "fast time" τ characterizes the distance between the trajectories φ_{i0}. Next we set $A_1 < A_2$ and $x_{10} - x_{20} = \text{const} > 0$, therefore, the trajectories $x = \varphi_{10}$ and $x = \varphi_{20}$ intersect at a point (x^*, t^*). We assume that the amplitude and phase corrections $S_i(\tau)$, $\varphi_{i1}(\tau)$ are such that

$$S_i \to 0 \quad \text{as} \quad \tau \to \pm\infty, \tag{19}$$

$$\varphi_{i1} \to 0 \quad \text{as} \quad \tau \to -\infty,$$
$$\varphi_{i1} \to \varphi_{i1}^{\infty} = \underset{i}{\text{const}} \quad \text{as} \quad \tau \to +\infty \tag{20}$$

with an exponential rate.

To find $S_i(\tau)$, $\varphi_{i1}(\tau)$ we should calculate the weak expansions for Q_i, P_i, and K_4. It is easy to check that

$$u = \varepsilon a_1 \sum_{i=1}^{2} \frac{G_i}{\beta_i}\delta(x - \varphi_i) + O_{\mathcal{D}'}(\varepsilon^3). \tag{21}$$

Here and in what follows we use the notation

$$a_k = \int_{-\infty}^{\infty} \omega^k(\eta)\, d\eta, \quad a_k^{(l)} = \int_{-\infty}^{\infty} \left(\frac{d^l\omega}{d\eta^l}\right)^k d\eta, \quad k \ge 1, \quad l \ge 1. \tag{22}$$

Next we take into account that $S_i(\tau)$ vanish exponentially fast as $|\varphi_1 - \varphi_2|$ grows, thus, the main contribution gives the point (x^*, t^*). We write

$$\varphi_{i0} = x^* + V_i(t - t^*) = x^* + \varepsilon\frac{V_i}{\dot\psi_0}\tau \text{ and } \varphi_i = x^* + \varepsilon\chi_i,\ \chi_i = V_i\tau/\dot\psi_0 + \varphi_{i1}, \tag{23}$$

where $\dot\psi_0 = \beta_1(V_2 - V_1)$. It remains to apply the formula

$$f(\tau)\delta(x - \varphi_i) = f(\tau)\delta(x - x^*) - \varepsilon\chi_i f(\tau)\delta'(x - x^*) + O_{D'}(\varepsilon^2), \tag{24}$$

which holds for each φ_i of the form (23) with slowly increasing χ_i and for $f(\tau)$ from the Schwartz space. Moreover, the second term in (24) is $O_{D'}(\varepsilon)$. Thus, under the assumptions (19) we can modify (21) to the final form:

$$u = \varepsilon a_1 \sum_{i=1}^{2} \frac{A_i}{\beta_i}\delta(x - \varphi_i) + \varepsilon a_1 \sum_{i=1}^{2} \frac{S_i}{\beta_i}\{\delta(x - x^*) - \varepsilon\chi_i\delta'(x - x^*)\} + O_{\mathcal{D}'}(\varepsilon^3). \tag{25}$$

Concerning nonlinear terms let us note that all the products of the waves with different phases are concentrated near the point (x^*, t^*). Thus, for any smooth function $F = F(z_1, \dots, z_4)$ we write the corresponding weak expansion:

$$\begin{aligned} F\left(u_{k_0,k_1}, \dots, \left(\varepsilon\frac{\partial}{\partial x}\right)^3 u_{k_0,k_1}\right) &= \varepsilon\Big\{\sum_{i=1}^{2} \frac{a_{F,i}^{(0)}}{\beta_i}\delta(x - \varphi_i) + \frac{1}{\beta_2}\mathcal{R}_F^{(0)}\delta(x - x^*)\Big\} \\ &\quad - \varepsilon^2\Big\{\sum_{i=1}^{2} \frac{a_{F,i}^{(1)}}{\beta_i^2}\delta'(x - \varphi_i) + \Big(\frac{\chi_2}{\beta_2}\mathcal{R}_F^{(0)} + \frac{1}{\beta_2^2}\mathcal{R}_F^{(1)}\Big)\delta'(x - x^*)\Big\} + O_{\mathcal{D}'}(\varepsilon^3). \end{aligned} \tag{26}$$

Here

$$a_{F,i}^{(n)} = \int_{-\infty}^{\infty} \eta^n F\left(A_i\omega(\eta), \dots, A_i\beta_i^3\omega'''(\eta)\right) d\eta, \quad n = 0,1, \tag{27}$$

$$\begin{aligned} \mathcal{R}_F^{(n)} = \int_{-\infty}^{\infty} \eta^n \{ & F(G_1\omega(\eta_{12}) + G_2\omega(\eta), \dots, G_1\beta_1^3\omega'''(\eta_{12}) + G_2\beta_2^3\omega'''(\eta)) \\ & - F(A_1\omega(\eta_{12}), \dots, A_1\beta_1^3\omega'''(\eta_{12})) \\ & - F(A_2\omega(\eta), \dots, A_2\beta_2^3\omega'''(\eta))\} d\eta, \end{aligned} \tag{28}$$

where

$$\eta_{12} = \theta\eta - \sigma, \quad \theta = \frac{\beta_1}{\beta_2}, \quad \sigma = \beta_1(\varphi_1 - \varphi_2)/\varepsilon. \tag{29}$$

We added the term $O_{\mathcal{D}'}(\varepsilon^2)$ to the right-hand sides of (25), (26) in order to calculate the time derivative with the precision mod $(O_{\mathcal{D}'}(\varepsilon^2))$ since

$$\frac{\partial}{\partial t} f(\tau(t,\varepsilon)) = \frac{\dot\psi_0}{\varepsilon}\frac{\partial}{\partial \tau} f(\tau).$$

Calculating weak expansions for all terms and substituting them into (4) we obtain linear combinations of $\delta'(x-\varphi_i)$, $i=1,2$, $\delta(x-x^*)$, and $\delta'(x-x^*)$. Therefore, we pass to the following 8×8 system of model equations:

$$V_i\, a^{(0)}_{Q_j,i} - a^{(0)}_{P_j,i} + \frac{a^{(1)}_{K_j,i}}{\beta_i} = 0, \quad i=1,2, \quad j=k_0,k_1, \tag{30}$$

$$\dot\psi_0 \frac{d}{d\tau}\mathcal{R}^{(0)}_{Q_j} + \mathcal{R}^{(0)}_{K_j} = 0, \quad j=k_0,k_1, \tag{31}$$

$$\dot\psi_0 \frac{d}{d\tau}\{\sum_{i=1}^{2}\varphi_{i1}\frac{a^{(0)}_{Q_j,i}}{\beta_i} + \frac{\chi_2}{\beta_2}\mathcal{R}^{(0)}_{Q_j} + \frac{1}{\beta_2^2}\mathcal{R}^{(1)}_{Q_j}\} - \frac{1}{\beta_2}\mathcal{R}^{(0)}_{P_j} + \frac{\chi_2}{\beta_2}\mathcal{R}^{(0)}_{K_j} + \frac{1}{\beta_2^2}\mathcal{R}^{(1)}_{K_j} = 0, \quad j=k_0,k_1. \tag{32}$$

Note that the property $a^{(0)}_{K_4,i}=0$ has been used here essentially.

Lemma 1. The algebraic equations (30) imply again the relations (3), (10) between A_i, β_i, and V_i.
For the proof it is enough to note that the function $A_i\,\omega(\beta_i(x-V_i t)/\varepsilon)$ is the exact classical solution of (1), thus it satisfies all conservation and balance laws associated with (1), see [6] for detail.

Furthermore, the system (31) contains two functional equations if $k_1<4$ or the functional equation and the ordinary equation if $k_1=4$. The system allows to define $S_i=S_i(\sigma)$, $i=1,2$.

Let us simplify the equations (32). We take into account the equalities (31) and the following consequences of the definitions (23), (29) of χ_i and σ:

$$\chi_2 = \frac{V_2}{\dot\psi_0}\tau + \varphi_{21}, \qquad \varphi_{11} = \varphi_{21} + \frac{1}{\beta_1}(\tau+\sigma). \tag{33}$$

Then the system (32) can be transformed to the form

$$\dot\psi_0 q_j\frac{d\varphi_{21}}{d\tau} + \frac{\dot\psi_0}{\beta_2}\frac{d}{d\tau}\{\sigma\frac{a^{(0)}_{Q_j,1}}{\theta^2} + \mathcal{R}^{(1)}_{Q_j}\} = f_j, \quad j=k_0,k_1, \tag{34}$$

where

$$q_j = \beta_2 r_j + \mathcal{R}^{(0)}_{Q_j},\ r_j = \sum_{i=1}^{2}\frac{a^{(0)}_{Q_j,i}}{\beta_i}, \qquad f_j = \mathcal{R}^{(0)}_{P_j} - \frac{\mathcal{R}^{(1)}_{K_j}}{\beta_2} - V_2\mathcal{R}^{(0)}_{Q_j} - \dot\psi_0 a^{(0)}_{Q_j,1}\frac{\beta_2}{\beta_1^2}. \tag{35}$$

Now is clear that (34) contains two unknown functions, φ_{21} and σ. Moreover, this system can be transformed easily to the autonomous equation

$$\mathcal{L}_{k_0,k_1}\frac{d\sigma}{d\tau} = \mathcal{F}_{k_0,k_1}, \tag{36}$$

where $\mathcal{F}_{k_0,k_1} = q_{k_1} f_{k_0} - q_{k_0} f_{k_1}$,

$$\mathcal{L}_{k_0,k_1} = \left(q_{k_1}p_{k_0} - q_{k_0}p_{k_1}\right)\frac{\dot\psi_0}{\beta_2}, \qquad p_j = a^{(0)}_{Q_j,1}\frac{\beta_2^2}{\beta_1^2} + \frac{d\mathcal{R}^{(1)}_{Q_j}}{d\sigma}. \tag{37}$$

Taking into account our hypothesis (20) we supply (36) by the scattering-type condition

$$\sigma/\tau \to -1 \quad \text{as} \quad \tau\to -\infty. \tag{38}$$

The behavior of the problem (36), (38) solution describes the scenario of the wave collision: if $\sigma/\tau\to -1$ when $\tau\to\infty$, then the waves interact elastically, each other behavior of σ means another scenario of collision.

2.1 Analysis of the Model Equations

In order to simplify the analysis let us assume:

$$k_0 = 1, \quad \theta << 1. \tag{39}$$

The last hypothesis allows to linearize the cumbrous formulas for the coefficients of the equations (31), (36). To do it let us introduce firstly normalized corrections $\varkappa_i = \varkappa_i(\tau)$ of the amplitudes,

$$S_1 = c\beta_1\beta_2^{\gamma-1}\theta^\gamma\varkappa_1, \quad S_2 = c\beta_2^\gamma\theta^\gamma\varkappa_2, \tag{40}$$

where the constant c and γ are defined in (10). Next we note that the first assumption (39) and the equation (31) for $j=1$ imply the equality

$$\varkappa_1+\varkappa_2=0. \tag{41}$$

Furthermore, let us consider a monoid

$$M_{\{m\}}\left(u_{k_0,k_1}\right) = (u_{k_0,k_1})^{m_0}\times\ldots\times\left((\varepsilon\frac{\partial}{\partial x})^3 u_{k_0,k_1}\right)^{m_3} \tag{42}$$

with integers $m_i\geq 0$. We apply (39) - (41) to (28) and assume the fulfilment of (19), (20). Then we obtain the following asymptotics for the function (42):

$$\mathcal{R}^{(n)}_{M,\{m\}} = (c\beta_2^{\gamma})^{|m|}\beta_2^{m'}\theta^{\gamma}\{m_0\mathcal{M}^{(n,1)}_{\{m\}}-|m|\varkappa_1\mathcal{M}^{(n,2)}_{\{m\}}\} + O_{\mathcal{S}}(\varepsilon^{2\gamma}), \tag{43}$$

$$\mathcal{M}^{(n,1)}_{\{m\}}=\int_{-\infty}^{\infty}\eta^n\,\omega_{12}\,\omega^{m_0-1}(\omega')^{m_1}(\omega'')^{m_2}(\omega''')^{m_3}\,d\eta,$$

$$\mathcal{M}^{(n,2)}_{\{m\}}=\int_{-\infty}^{\infty}\eta^n\,\omega^{m_0}\times\ldots\times(\omega''')^{m_3}\,d\eta,$$

where $|m|=\sum_{l=0}^{3}m_l$, $m'=\sum_{l=1}^{3}l\,m_l$, and we use the following definition of the smallness:

Definition 4. A function $v(\tau,\theta)$ is said to be of the value $O_{\mathcal{S}}(\theta^{\alpha})$ if there exists a function $f(\tau)\geq 0$ from the Schwartz space $\mathcal{S}$ such that the inequality

$$|v(\tau,\theta)|\leq\theta^{\alpha}f(\tau)$$

holds uniformly in $\tau\in\mathbb{R}$.

Let us turn to (31) for $k_1=2$ or 3. They are algebraic equations of the order 2 and 5 respectively, therefore applying (43) we obtain

$$\varkappa_1=\mathcal{K}_{k_1}(\sigma)+O_{\mathcal{S}}(\theta^{\gamma}),\quad k_1=2,3, \tag{44}$$

where

$$\mathcal{K}_2(\sigma)=\frac{\mathcal{M}^{(0,1)}_{\{20\}}}{a_2},\qquad \mathcal{K}_3(\sigma)=\frac{20}{17}\mathcal{M}^{(0,1)}_{\{50\}}/a_5. \tag{45}$$

Here we use the equality $a_2^{(1)}=3a_5/4$ [6] and write $\{20\}$, $\{010\}$ instead of $\{2000\}$, $\{0100\}$ and so on.

Under the assumption (20) the convolutions $\mathcal{M}^{(n,1)}_{\{m\}}$ belong to the Schwartz space $\mathcal{S}$. Thus $\varkappa_i$ satisfy the hypothesis (19) in the cases $k_1=2,3$. Moreover, in both cases $\varkappa_i=\varkappa_i(\sigma)$ are even functions of the same order $O(1)$ in the C-sense.

In the case $k_1=4$ we have the differential equation

$$\frac{d}{d\tau}\{c_1\varkappa_1-\mathcal{K}_4\} =\frac{3}{10}\int_{-\infty}^{\infty}(\omega'(\eta))^4\omega'(\eta_{12})d\eta+O_{\mathcal{S}}(\theta^{\gamma}), \tag{46}$$

where

$$c_1=\mathcal{M}^{(0,2)}_{\{320\}}-\frac{2}{7}a_8-\frac{3}{5^3}a_2^{(2)},\qquad \mathcal{K}_4=\frac{2}{7}\mathcal{M}^{(0,1)}_{\{80\}}-\frac{3}{5}\mathcal{M}^{(0,1)}_{\{320\}}+O_{\mathcal{S}}(\theta^{\gamma}). \tag{47}$$

The combination of convolutions $\mathcal{K}_4\in\mathcal{S}$ is an even function with respect to σ, whereas the right-hand side is an odd one. Thus, under the hypothesis

$$\frac{d\sigma}{d\tau}\quad\text{is an even function} \tag{48}$$

the last term in (46) disappears after the integration over $\mathbb{R}$. Therefore, $\varkappa_1$ turns out again to be an even function which satisfies the hypothesis (19).

Similar analysis of the equation (36) shows that its structure is the same for each k_1,

$$\mathcal{L}_{1,k_1}=1+O_{\mathcal{S}}(\theta),\qquad \mathcal{F}_{1,k_1}=-1+O_{\mathcal{S}}(\theta^{n_{k_1}\gamma}), \tag{49}$$

where $n_2=1$, $n_3=n_4=1/2$. Obviously, for sufficiently small θ the problem (36), (38) is solvable and $\sigma/\tau\to -1$ as $\tau\to\infty$.

Finally, to check the property (48) we note that all Q_j, P_j, calculated for the single solitary wave, are the even functions, whereas K_4 is the odd function. Thus, all $\mathcal{R}^{(0)}_{Q_j}$, $\mathcal{R}^{(0)}_{P_j}$ and $\mathcal{R}^{(1)}_{4}$ are even functions with respect to σ. On the contrary, all $\mathcal{R}^{(1)}_{Q_j}$, $\mathcal{R}^{(1)}_{P_j}$ and $\mathcal{R}^{(0)}_{4}$ are odd functions. This and representation (35), (37) imply the fulfilment of (48). The last step of the construction is the return to the phase corrections φ_{i1}. In view of (33), (34) it is obvious that the last assumption of the form (20) is justified. This implies the proposition

Theorem 3. Under the assumption (39) the asymptotic solution in the sense of Definition 3 exists for each k_1 and describes the elastic collision of the solitary waves.

Now let us compare the asymptotics which are possible for various choice of k_1. Firstly let us note that the replacement of one $\varphi_{i1}^{(k_1)}$ by another $\varphi_{i1}^{(k'_1)}$ implies a small correction in the sense of Definition 2. Indeed,

$$(\omega(\beta_i \frac{x-\varphi_{i0}-\varepsilon\varphi_{i1}^{(k_1)}}{\varepsilon}) - \omega(\beta_i \frac{x-\varphi_{i0}-\varepsilon\varphi_{i1}^{(k'_1)}}{\varepsilon}), \psi(x)) = a_1 \frac{\varepsilon}{\beta_i}(\delta(x-\varphi_{i0}-\varepsilon\varphi_{i1}^{(k_1)}) - \delta(x-\varphi_{i0}- \varepsilon\varphi_{i1}^{(k'_1)}), \psi(x)) + O(\varepsilon^3) = O(\varepsilon^2).$$

The correction is beyond the rang of the weak asymptotics accuracy.

Furthermore, according to (45), (46) the shape of the amplitudes S_i depends on the selection of the conservation/balance laws. The difference between them is $O(1)$ in the C-sense and $O_{\mathcal{D}'}(\varepsilon)$ in the sense of Definition 2. We stress that such phenomena of nonuniqueness is immanently intrinsic to weak asymptotics. In particular, it is possible to change the shape of the amplitudes adding to (2) arbitrarily small in the weak sense functions [1]. However, this effect is concentrated within $O(\varepsilon^{1-\nu})$-neighborhood of the time-instant t^* of the interaction, $\nu > 0$. Thus it is small in the $\mathcal{D}'(\mathbb{R}^2_{x,t})$ sense. It implies the following

Definition 5. Let functions $u_1(x,t,\varepsilon)$ and $u_2(x,t,\varepsilon)$ satisfy the problem (1), (16) in the sense of Definition 3. Then these functions are said to be asymptotically equivalent if the relation

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{\tilde{u}_1(x,t,\varepsilon) - \tilde{u}_2(x,t,\varepsilon)\}\psi(x,t)dx\,dt = O(\varepsilon^2)$$

holds for any test function $\psi \in \mathcal{D}(\mathbb{R}^2)$. Here $\tilde{u}_k$ are continuations of u_k for $t < 0$ as noninteracting solitary waves.

Now we can formulate the main result of the paper:

Theorem 4. Under the assumptions (39) the weak asymptotic solutions u_{1,k_1} and u_{1,k'_1} of the problem (1), (16) are asymptotically equivalent for all $k_1, k'_1 \in \{2,3,4\}$.

As a conclusion we propose the hypothesis that the similar to Theorem 2 statement should be true for all pairs $\{k_0, k_1\}$ and for all higher balance laws. Of course there is the assumption that the "production" K_j, $j \geq 4$, is an odd function in the same sense as K_4. We guess also that the same is valid for bigger number of colliding solitons. Numerical simulations [3] verify this conjecture, see Fig. 3.

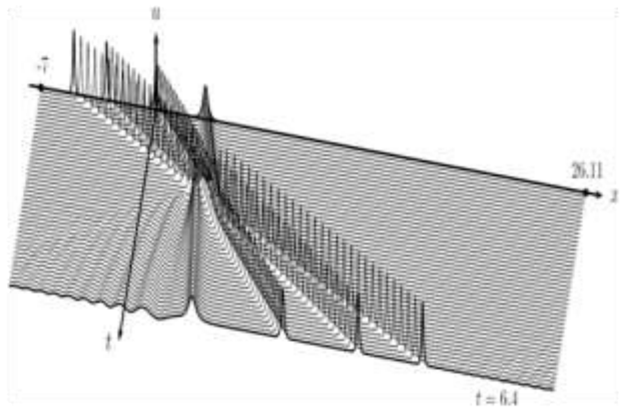

Fig. 3 Evolution of 4 solitons with $\beta_1 = 0.7, \beta_2 = 1.25$, $\beta_3 = 1.62$ and $\beta_4 = 1.83$ for $\varepsilon = 0.1$.

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