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Explicit Form of First Integral and Limit Cycles for a Class of Planar Systems

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Abstract: In this paper we introduce an explicit form for the first integral in the way to study the integrability and consequently the possibility of existence or nonexistence of limit cycles of a certain class of planar differential systems. A concrete example exhibiting the applicability of the result is introduced.

Key words: first intergal, planar system, limit cycle.

1. Introduction

In this paper, we study the integrability and then the possibility of the existence or nonexistence of periodic orbits of class of planar differential systems defined on the real plane by

$$\dot{x} = F(x,y) \qquad (1.1)$$
$$\dot{y} = G(x,y)$$

where F, G are analytic functions as elements in the ring of analytic functions in two variable $R[x,y]$. The dot denotes differentiation with respect to the independent variable t usually called time, that is $\dot{x} = \frac{dx}{dt}$. These differential systems are mathematical models arise in many fields of application like biology, physics and engineering, see [1]. The study of the dynamics of 1.1 strongly depends on the existence stability properties, number and location of special solution such as singular points and non-constant isolated periodic solutions. In particular, if an attracting non-constant isolated periodic solution exists, then it dominates the dynamics of the system 1.1 in an open connected subset of the plane, its region of attraction, such periodic solution called *limit cycles*. In some cases such a region of attraction can extend to cover the whole plane, with the exception of a singular point. In such a case the limit cycle is unique and dominates the system's dynamics. Uniqueness of limit cycles have been extensively studied in many books and articles, see for example [2], [3], [4], [5] and references therein.

The vector field associated to systems 1.1 will be denoted by

$$X(x,y) = (F(x,y), G(x,y))$$

Notes that the ordinary differential equation

$$F(x,y)dx - G(x,y)dy = 0$$

is just the differential equation of that orbits of system 1.1.

This work is related to the integrability problem which is defined as the problem of finding a first integral for a planar differential system and determining the functional class it must belong to. Then we discuss the possibility of existence and non-existence of limit cycles. Recall that a first integral $H(x,y)$ of system 1.1 is a function defined on some open set U of R^2, non locally constant and which satisfies the following partial differential equation:

$$F(x,y)\frac{\partial H}{\partial x}(x,y) + G(x,y)\frac{\partial H}{\partial y}(x,y) = 0$$

In the qualitative theory of planar dynamical systems to study the number and location of limit cycles is one

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of most important topics which is related to the second part of the unsolved Hilbert 16th problem see [6], [7], [8], [9], [10], [11]. There is a huge literature about limit cycles, most of them deal essentially with their detection, their number and their stability and rare are papers concerned by giving them explicitly see [12]. In fact, for differential systems defined on the plane R^2, the existence of a first integral determines their phase portrait see [13]. Here we are interested in finding an explicit form for a first integral of the differential system given by

$$\dot{x} = \sum_{i=1}^{N} x^{\alpha_i} y^{\beta_i} P_{n_i}(x,y) + \sum_{i=1}^{T} x^{\sigma_i+1} y^{\rho_i} R_{k_i}(x,y)$$
$$\dot{y} = \sum_{i=1}^{M} x^{\mu_i} y^{\upsilon_i} Q_{m_i}(x,y) + \sum_{i=1}^{T} x^{\sigma_i} y^{\rho_i+1} R_{k_i}(x,y) \tag{1.2}$$

where $\alpha_j + \beta_j + n_j = \mu_j + \upsilon_j + m_j = l$ and $\sigma_j + \rho_j + k_j = t$ for all positive integer j, and m_j, n_j, $k_j \in Z^+$ but α_j, β_j, μ_j, υ_j, σ_j, $\rho_j \in R$.

In section 2 we give the main results with the proofs. In section 3, we present a concrete example to illustrate the applicability of the result.

2. Main Results

In this section we concern with construction of an explicit form for a first integral of two dimensional autonomous of real differential system 1.2. Our first result is the following Theorem.

Theorem 1. If $h(\theta) \neq 0$ for all θ, then the function

$$H(x,y) = (x^2+y^2)^{\frac{l-t-1}{2}} Exp[(1+t-l)\int^{\arctan\frac{y}{x}} A(\omega)d\omega] \tag{2.1}$$

$$+ (1+t-l)\int^{\arctan\frac{y}{x}} B(\omega) Exp[(1+t-l)\int^{\omega} A(\theta)d\theta]d\omega \tag{2.2}$$

form the first integral for the differential system 1.2. Moreover the system has no limit cycle surrounding the origin, where

$$A(\theta) = \frac{f(\theta)}{h(\theta)}$$

$$B(\theta) = \frac{g(\theta)}{h(\theta)}$$

$$f(\theta) = \sum_{i=1}^{N} \cos^{\alpha_i+1}\theta \sin^{\beta_i}\theta P_{n_i}(\cos\theta, \sin\theta) + \sum_{i=1}^{M} \cos^{\mu_i}\theta \sin^{\upsilon_i+1}\theta Q_{m_i}(\cos\theta, \sin\theta) \tag{2.3}$$

$$g(\theta) = \sum_{i=1}^{T} \cos^{\sigma_i}\theta \sin^{\rho_i}\theta R_{k_i}(x,y)$$

$$h(\theta) = -\sum_{i=1}^{N} \cos^{\alpha_i}\theta \sin^{\beta_i+1}\theta P_{n_i}(\cos\theta, \sin\theta) + \sum_{i=1}^{M} \cos^{\mu_i+1}\theta \sin^{\upsilon_i}\theta Q_{m_i}(\cos\theta, \sin\theta)$$

Proof. In polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, system 1.2 may be written in the form

$$\dot{r} = \cos\theta\left[\sum_{i=1}^{N}(r\cos\theta)^{\alpha_i}(r\sin\theta)^{\beta_i}P_{n_i}(r\cos\theta, r\sin\theta) + \sum_{i=1}^{T}(r\cos\theta)^{\sigma_i+1}(r\sin\theta)^{\rho_i}R_{k_i}(r\cos\theta, r\sin\theta)\right] + \sin\theta\left[\sum_{i=1}^{M}(r\cos\theta)^{\mu_i}(r\sin\theta)^{\upsilon_i}Q_{m_i}(r\cos\theta, r\sin\theta) + \sum_{i=1}^{T}(r\cos\theta)^{\sigma_i}(r\sin\theta)^{\rho_i+1}R_{k_i}(r\cos\theta, r\sin\theta)\right] = r^l f(\theta) + r^{t+1}g(\theta) \quad (2.4)$$

and

$$\dot{\theta} = \frac{-\sin\theta}{r}\left[\sum_{i=1}^{N}(r\cos\theta)^{\alpha_i}(r\sin\theta)^{\beta_i}P_{n_i}(r\cos\theta, r\sin\theta) + \sum_{i=1}^{T}(r\cos\theta)^{\sigma_i+1}(r\sin\theta)^{\rho_i}R_{k_i}(r\cos\theta, r\sin\theta)\right] + \frac{\cos\theta}{r}\left[\sum_{i=1}^{M}(r\cos\theta)^{\mu_i}(r\sin\theta)^{\upsilon_i}Q_{m_i}(r\cos\theta, r\sin\theta) + \sum_{i=1}^{T}(r\cos\theta)^{\sigma_i}(r\sin\theta)^{\rho_i+1}R_{k_i}(r\cos\theta, r\sin\theta)\right] = r^{l-1}h(\theta) \quad (2.5)$$

where f, g and h are the trigonometric defined above, in the formula 2.1.

Taking the coordinates θ as a new independent variable, equations 2.4 and 2.5 will be in the form

$$\frac{dr}{d\theta} = A(\theta)r + B(\theta)r^{t-l+2} \quad (2.6)$$

where $A(\theta) = \frac{f(\theta)}{h(\theta)}$, $B(\theta) = \frac{g(\theta)}{h(\theta)}$, which is a Bernoulli equation.

If we introduce the standard change of variables $\rho = r^{l-t-1}$, we obtain the following linear differential equation

$$\frac{d\rho}{d\theta} = (l-t-1)[A(\theta)\rho + B(\theta)]$$

whose general solution is given by

$$\rho(\theta) = Exp[(l-t-1)^{\theta}\int A(\omega)d\omega] \times \left[C + (l-t-1)^{\theta}\int B(\omega)Exp[(1+t-l)^{\omega}\int A(u)du]d\omega\right]$$

where C is an arbitrary real number.

If we replace the arbitrary constant C by $H(x,y)$, which is the first integral, and rearrange the terms we obtain the desired form given in the statement of the Theorem.

Since all the points $(x,0)$ and $(0,y)$ for all x, y are singular points, then the system can not have a limit cycle surrounding the origin. This completes the proof of the Theorem. □

Theorem 2. If the distinct real roots of the equation $h(\theta) = 0$, are $\theta_1, \theta_2, \ldots, \theta_n$, then for the region where $\theta \notin \{\theta_1, \theta_2, \ldots, \theta_n\}$, the function $H(x,y)$ defined in 2.1 is the first integral for the system 1.2.

Proof. Direct. □

Theorem 3. If the distinct real roots of the equation $h(\theta) = 0$, are $\theta_1, \theta_2, \ldots, \theta_n$, then the straight lines $y = (\tan\,\theta_i)x$, $i = 1,2,\ldots,n$ (the $y-axis$ for $\theta_i = \frac{\pi}{2}$) are particular solutions for the system and hence the system has no periodic orbits and consequently no limit cycles surrounding the origin.

Proof. We have $h(\theta) = 0,\ i = 1,2,\ldots,n$.

Hence $\theta(t_i) = \theta_i$, for $i = 1,2,\ldots,n$, are initial conditions for which the straight lines $y = (\tan\,\theta_i)x$, $i = 1,2,\ldots,n$ (the $y-axis$ for $\theta_i = \frac{\pi}{2}$) are solutions since $h(\theta) = 0$, for all r.

Therefore $\dot{\theta}(t_i) = 0$, for $i = 1,2,\ldots,n$.

So the straight lines $y = (\tan\,\theta_i)x$, $i = 1,2,\ldots,n$ (the $y-axis$ for $\theta_i = \frac{\pi}{2}$) through the origin are invariant curves by the flow for the system.

This implies that the system has no periodic orbits surrounding the origin.

This completes the proof of the Theorem. □

Corollary 1. If $h(\theta) = 0$ for some value of θ and the system has only one singular point at the origin then the system has no periodic orbits.

Theorem 4. If $h(\theta) = 0$ for all $\theta \in R$, then the function $H(x, y) = \frac{y}{x}$, is a first integral for the system 1.2 and consequently the system has no periodic orbit.

Proof. From formulas 5 and 7 follows that $\dot{\theta} = 0$.

Hence $\theta(t) = c$, then $\frac{y(t)}{x(t)} = c$.

Therefore the straight lines $y = cx$ through the origin are invariant curves by the flow of the system.

This implies that $H(x, y) = \frac{y}{x}$ is a first integral for the system 1.2.

Since all the straight lines through the origin are found by trajectories then the system has no periodic orbits, and consequently no limit cycles.

This completes the proof of the Theorem. □

3. Application

In this section a concrete example illustrating the applicability of the Theorems is introduced.

3.1. Example. Consider the system

$$\begin{aligned} \dot{x} &= x^{\alpha} y^{\beta} + x^{\alpha+1} y^{\beta-3}(x - y) \\ \dot{y} &= x^{\alpha} y^{\beta} + x^{\alpha} y^{\beta-2}(x - y) \end{aligned} \tag{3.1}$$

where $\alpha, \beta \in R,\ \alpha\beta \neq 0$. This system is the system 1.2 regarding $M = N = T = 1$, $P_{n_i}(x, y) = Q_{m_i}(x, y) \equiv 1$, $R_{k_i}(x, y) = R_1(x, y) = x - y$, $\mu_i = \alpha_i = \alpha$, $\upsilon_i = \beta_i = \beta$, $\sigma_i = \alpha_i = \alpha$, $\rho_i = \beta_i - 3 (= \beta - 3)$, $l = \alpha_j + \beta_j + n_j = \mu_j + \upsilon_j + m_j$, $n_j = m_j = 0$, $t = \sigma_j + \rho_j + k_j$, $k_j = 1$.

$$\begin{aligned} f(\theta) &= \cos^{\alpha+1}\theta \sin^{\beta}\theta P_0(\cos\theta, \sin\theta) \\ &\quad + \cos^{\alpha}\theta \sin^{\beta+1}\theta Q_0(\cos\theta, \sin\theta) \end{aligned}$$

since $P_0(\cos\theta, \sin\theta) = Q_0(\cos\theta, \sin\theta) = 1$, then

$$f(\theta) = \cos^{\alpha+1}\theta \sin^{\beta}\theta + \cos^{\alpha}\theta \sin^{\beta+1}\theta$$

$$g(\theta) = \cos^{\alpha}\theta \sin^{\beta-3}\theta(\cos\theta - \sin\theta)$$

$$h(\theta) = \cos^{\alpha}\theta \sin^{\beta}\theta(\cos\theta - \sin\theta)$$

Hence

$$A(\theta) = \frac{f(\theta)}{h(\theta)} = \frac{\cos\theta + \sin\theta}{-\sin\theta + \cos\theta}$$

$$B(\theta) = \frac{g(\theta)}{h(\theta)} = \frac{1}{\sin^3\theta}$$

Therefore, we have

$$Exp^{\omega} \int A(\theta)d\theta = \sin\omega - \cos\omega$$

Hence

$$\int^{\arctan\frac{y}{x}} B(\omega) Exp[-^{\omega} \int A(\theta)d\theta] d\omega$$

$$= -\frac{x}{y} + \frac{1}{2}\frac{x^2 + y^2}{y^2}$$

Then we obtain the following explicit form of the first integral of the system 9,

$$H(x, y) = y - x + \frac{x}{y} - \frac{1}{2}\left(\frac{x}{y}\right)^2 - \frac{1}{2}$$

From which we have

$$\frac{\partial H}{\partial x} = -1 + \frac{1}{y} - \frac{x}{y^2} \quad \text{and} \quad \frac{\partial H}{\partial y} = 1 - \frac{x}{y^2} + \frac{x^2}{y^3}$$

Therefore,

$$\frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\dot{y} = 0$$

From $h(\theta) = 0$, we have:

$\sin\theta = \cos\theta$, $\cos\theta = 0$, $\sin\theta = 0$

Hence the zeros of $h(\theta)$ are $\theta = 0, \frac{\pi}{2}, \frac{3\pi}{2}, \pi, \frac{\pi}{4}, \frac{5\pi}{4}$. So the distinct zeros are $0, \frac{\pi}{4}, \frac{\pi}{2}$ Therefore the straight lines $y = (\tan\theta_i)x$, $\theta_i \in \left\{0, \frac{\pi}{4}, \frac{\pi}{2}\right\}$ and (the $y-axis$ for $\theta = \frac{\pi}{2}$) through the origin are invariant curves by the flow of the system. On the other hand the system 1.2 has no singular points in the region $\theta \notin \left\{0, \frac{\pi}{4}, \frac{\pi}{2}\right\}$ except the origin. From this we induce that the system has no periodic orbits and consequently no limit cycle.

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