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Randomness of Lacunary Statistical Acceleration Convergence of χ^2 over p –Metric Spaces Defined by Orlicz Function

Deepmala[1], N. Subramanian[2], Lakshmi Narayan Mishra[3]

1. SQC and OR Unit, Indian Statistical Institute, 203 B. T. Road, Kolkata, 700 108,, India

2. Department of Mathematics, SASTRA University, Thanjavur-613 401, India

3. Department of Mathematics, National Institute of Technology, Silchar 788 010, District - Cachar (Assam), India

Abstract: In this article the notion of Randomness of Lacunary statistical acceleration convergence of χ^2 over $p-$ metric spaces defined by sequence of Orlicz has been introduced and some theorems related to that concept have been established using the four dimensional matrix transformations. Author's construct with new definition's and also new statement of theorems of proves are formulated.

Key words: analytic sequence, double sequences, χ^2 space, Musielak-Orlicz function, random $p-$ metric space, lacunary sequence, statistical convergence, converging faster, converging at the same rate, acceleration field, double natural density

1. Introduction

The faster convergence of sequences particularly the acceleration of convergence of sequence of partial sums of series via linear and nonlinear transformations are widely used in finding solutions of mathematical as well as different scientific and engineering problems. The problem of acceleration convergence often occurs in numerical analysis. To accelerate the convergence, the standard interpolation and extrapolation methods of numerical mathematics are quite helpful. It is useful to study about the acceleration of convergence methods, which transform a slowly converging sequence into a new sequence, converging to the same limit faster than the original sequence. The speed of convergence of sequences is of the central importance in the theory of sequence transformation.

Corresponding author: Deepmala, Prof., Ph.D., research fields: optimization; approximation theory; nonlinear analysis, integral equations; modeling etc. E-mail: dmrai23@gmail.com, deepmaladm23@gmail.com.

The concept of statistical convergence play a vital role not only in pure mathematics but also in other branches of science involving mathematics, especially in information theory, computer science, biological science, dynamical systems, geographic information systems, population modeling, and motion planning in robotics.

The notion of statistical convergence was introduced by Fast [1] and Schoenberg [2] independently. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Later on it was further investigated by Fridy, $\check{S}$al$\acute{a}$t, cCakalli, Maio and Kocinac , Miller , Maddox , Leindler , Mursaleen and Alotaibi , Mursaleen and Edely , and many others. In the recent years, generalizations of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on Stone-$\check{C}$ech compactification of the natural numbers. Moreover statistical convergence is closely related to the concept of convergence in probability, (see [3]).

The notion of statistical convergence depends on the density of subsets of $\mathbb{N}$. A subset of $\mathbb{N}\times\mathbb{N}$ is said to have density $\delta(E)$ if

$$\delta(E) = lim_{r,s\to\infty} \frac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\chi^2\, E(mn) = 0$$

Throughout w,χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively.

We write w^2 for the set of all complex sequences (x_{mn}), where $m,n\in\mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces is found in Bromwich [3]. Later on, they were investigated by Hardy [4], Moricz [5], Moricz and Rhoades [6], Basarir and Solankan [7], Tripathy [8], Turkmenoglu [9], and many others.

Let (x_{mn}) be a double sequence of real or complex numbers. Then the series $\sum_{m,n=1}^{\infty} x_{mn}$ is called a double series. The double series $\sum_{m,n=1}^{\infty} x_{mn}$ give one space is said to be convergent if and only if the double sequence $(S_{mn})$is convergent, where

$$S_{mn} = \sum_{i,j=1}^{m,n} x_{ij}\ (m,n = 1,2,3,\dots).$$

A double sequence $x=(x_{mn})$is said to be double analytic if

$$sup_{m,n}|x_{mn}|^{\frac{1}{m+n}} < \infty.$$

The vector space of all double analytic sequences are usually denoted by Λ^2. A sequence $x=(x_{mn})$ is called double entire sequence if

$$|x_{mn}|^{\frac{1}{m+n}} \to 0 \text{ as } m,n\to\infty.$$

The vector space of all double entire sequences are usually denoted by Γ^2. Let the set of sequences with this property be denoted by Λ^2 and Γ^2 is a metric space with the metric

$$d(x,y) = sup_{m,n}\left\{|x_{mn}-y_{mn}|^{\frac{1}{m+n}} : m,n\colon 1,2,3,\dots\right\}, \quad (1.1)$$

for all $x=\{x_{mn}\}$ and $y=\{y_{mn}\} in \Gamma^2$. Let $\phi = \{finite\ sequences\}$.

Consider a double sequence $x=(x_{mn})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij}\,\delta_{ij}$ for all $m,n\in\mathbb{N}$,

$$\delta_{mn} = \begin{pmatrix} 0 & 0 & \dots 0 & 0 & \dots \\ 0 & 0 & \dots 0 & 0 & \dots \\ . & & & & \\ . & & & & \\ . & & & & \\ 0 & 0 & \dots 1 & 0 & \dots \\ 0 & 0 & \dots 0 & 0 & \dots \end{pmatrix}$$

with 1 in the $(m,n)^{th}$ position and zero otherwise.

A double sequence $x=(x_{mn})$ is called double gai sequence if $((m+n)!\,|x_{mn}|)^{\frac{1}{m+n}} \to 0$ as $m,n\to\infty$. The double gai sequences will be denoted by χ^2. An Orlicz function is a function $f\colon[0,\infty)\to[0,\infty)$ which is continuous, non-decreasing and convex with $f(0)=0, f(x)>0$, for $x>0$ and $f(x)\to\infty$ as $x\to\infty$. If convexity of Orlicz function f is replaced by $f(x+y)\le f(x)+f(y)$, then this function is called modulus function. An Orlicz function f is said to satisfy Δ^2- condition for allvalues u, if there exists $K>0$ such that $f(2u)\le Kf(u), u\ge 0$.

1.1. Lemma. Let f be an Orlicz function which satisfies Δ^2- condition and let $0<\delta<1$. Then for each $t\ge\delta$, we have $f(t)<K\delta^{-1}f(2)$ for some constant $K>0$.

A sequence $f=(f_{mn})$ of Orlicz function is called a Musielak-Orlicz function. A sequence $g=(g_{mn})$ defined by

$$g_{mn}(v) = sup\{|v|u-(f_{mn})(u): u\ge 0\}, m,n = 1,2,\cdots$$

is called the complementary function of a sequence of Musielak-Orlicz f. For a given sequence of Musielak-Orlicz function f, the Musielak-Orlicz sequence space t_f is defined as follows

$$t_f = \left\{x\in w^2 : I_f(|x_{mn}|)^{1/m+n} \to 0 as m,n\to\infty\right\},$$

where I_f is a convex modular defined by

$$I_f(x) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn}\,(|x_{mn}|)^{1/m+n}, x=(x_{mn})\in t_f.$$

2. Definition and Preliminaries

Let $n\in\mathbb{N}$ and X be a real vector space of dimension w, where $n\le w$. A real valued function

$d_p(x_1, \ldots, x_n) = \| (d_1(x_1, 0), \ldots, d_n(x_n, 0)) \|_p$ on X satisfying the following four conditions:

(i) $\| (d_1(x_1, 0), \ldots, d_n(x_n, 0)) \|_p = 0$ if and and only if $d_1(x_1, 0), \ldots, d_n(x_n, 0)$ are linearly dependent;

(ii) $\| (d_1(x_1, 0), \ldots, d_n(x_n, 0)) \|_p$ is invariant under permutation;

(iii) $\| (\alpha d_1(x_1, 0), \ldots, d_n(x_n, 0)) \|_p = |\alpha| \| (d_1(x_1, 0), \ldots, d_n(x_n, 0)) \|_p, \alpha \in \mathbb{R}$;

(iv) $d_p((x_1, y_1), (x_2, y_2) \cdots (x_n, y_n)) = (d_X(x_1, x_2, \cdots x_n)^p + d_Y(y_1, y_2, \cdots y_n)^p)^{1/p} for 1 \leq p < \infty$; (or)

(v) $d((x_1, y_1), (x_2, y_2), \cdots (x_n, y_n)) = \sup \{d_X(x_1, x_2, \cdots x_n), d_Y(y_1, y_2, \cdots y_n)\}$,

for $x_1, x_2, \cdots x_n \in X, y_1, y_2, \cdots y_n \in Y$ is called the $p-$ product metric of the Cartesian product of $n-$metric spaces is the $p-$norm of the n-vector of the norms of the $n-$sub spaces.

A trivial example of $p-$ product metric of $n-$ metric space is the $p-$ norm space is $X = \mathbb{R}$ equipped with the following Euclidean metric in the product space is the $p-$norm:

$$\| (d_1(x_1, 0), \ldots, d_n(x_n, 0)) \|_E = sup(|det(d_{mn}(x_{mn}, 0))|)$$

$$= \sup \left(\begin{vmatrix} d_{11}(x_{11}, 0) & d_{12}(x_{12}, 0) & \ldots & d_{1n}(x_{1n}, 0) \\ d_{21}(x_{21}, 0) & d_{22}(x_{22}, 0) & \ldots & d_{2n}(x_{1n}, 0) \\ & . & & . \\ & . & & . \\ & . & & . \\ d_{n1}(x_{n1}, 0) & d_{n2}(x_{n2}, 0) & \ldots & d_{nn}(x_{nn}, 0) \end{vmatrix} \right)$$

where $x_i = (x_{i1}, \cdots x_{in}) \in \mathbb{R}^n$ for each $i = 1,2, \cdots n$.

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the $p-$ metric. Any complete $p-$ metric space is said to be $p-$ Banach metric space.

2.1. Definition

A sequence space E_F of fuzzy numbers is said to be (i) solid (or normal) if $(Y_{mn}) \in E_F$ whenever $(X_{mn}) \in E_F$ and $\bar{d}(Y_{mn}, \bar{0}) \leq \bar{d}(X_{mn}, \bar{0})$ for all $m, n \in \mathbb{N}$. (ii) symmetric if $(X_{mn}) \in E_F$ implies $(X_{\pi(mn)}) \in E_F$ where π is a permutation of $\mathbb{N} \times \mathbb{N}$.

Let $K = \{k_1 < k_2 < \ldots\} \subseteq \mathbb{N}$ and E be a sequence space. A K-step space of E is a sequence space

$$\lambda_{mn}^E = \left\{ \left(X_{m_p n_p} \right) \in w^2 : \left(m_p n_p \right) \in E \right\}.$$

A canonical preimage of a sequence $\left\{ \left(x_{m_p n_p} \right) \right\} \in \lambda_K^E$ is a sequence $\{y_{mn}\} \in w^2$ defined as

$$y_{mn} = \begin{cases} x_{mn}, & \text{if } m, n \in E \\ 0, & \text{otherwise.} \end{cases}$$

A canonical preimage of a step space λ_K^E is a set of canonical preimages of all elements in λ_K^E, i.e. y is in canonical preimage of λ_K^E if and only if y is canonical preimage of some $x \in \lambda_K^E$.

2.2. Definition

A sequence space E_F is said to be monotone if E_F contains the canonical pre-images of all its step spaces.

By the convergence of a double sequence we mean the convergence on the Pringsheim sense that is, a double sequence $x = (x_{mn})$ has Prinsheim limit L (denoted by $P - \lim x = L$) provided that given $\epsilon > 0$ there exists $n \in bN$ such that $|x_{mn} - L| < \epsilon$ whenever $m, n > n$. We shall write more briefly as $P-$ convergent.

The double sequence $\theta_{rs} = \{(m_r, n_s)\}$ is called double lacunary sequence if there exist two increasing of integers such that

$$m_0 = 0, \varphi_r = m_r - m_{r-1} \to \infty \text{ as } r \to \infty \text{ and}$$

$$n_0 = 0, \varphi_s = n_s - n_{s-1} \to \infty \text{ as } s \to \infty.$$

Notations: $m_{rs} = m_r n_s, h_{rs} = \varphi_r \bar{\varphi}_s, \theta_{rs}$ is determined by

$$I_{rs} = \{(m, n): m_{r-1} < m \leq m_r and n_{s-1} < n \leq n_s\},$$

$$q_r = \frac{k_r}{k_{r-1}}, \bar{q}_s = \frac{n_s}{n_{s-1}} \text{ and } q_{rs} = q_r \bar{q}_s.$$

The notion of $\lambda-$ double gai and double analytic sequences as follows: Let $\lambda = (\lambda_{mn})_{m,n=0}^{\infty}$ be a strictly increasing sequences of positive real numbers tending to infinity, that is

$$0 < \lambda_{00} < \lambda_{11} < \cdots and \lambda_{mn} \to \infty as m, n \to \infty$$

and said that a sequence $X = (X_{mn}) \in w^2$ is $\lambda -$ convergent to 0, called a the $\lambda -$ limit of X, if $\mu_{mn}(X) \to 0 \; as \; m, n \to \infty$, where $\mu_{mn}(X) =$

$$\overline{d}\left(\frac{1}{\varphi_{rs}}\sum_{m\in I_{rs}}\sum_{n\in I_{rs}}\left(\Delta^{m-1}\lambda_{m,n} - \Delta^{m-1}\lambda_{m,n+1} - \Delta^{m-1}\lambda_{m+1,n} + \Delta^{m-1}\lambda_{m+1,n+1}\right)|X_{mn}|^{\frac{1}{m}+n}, \overline{0}\right).$$

The sequence $X = (X_{mn}) \in w^2$ is $\lambda -$ double analytic if $sup_{uv}|\mu_{mn}(X)| < \infty$. If $lim_{mn} X_{mn} = 0$ in the ordinary sense of convergence, then

$$lim_{mn}\; \overline{d}\left(\frac{1}{\varphi_{rs}}\sum_{m\in I_{rs}}\sum_{n\in I_{rs}}\begin{pmatrix}\Delta^{m-1}\lambda_{m,n}\\ -\Delta^{m-1}\lambda_{m,n+1}\\ -\Delta^{m-1}\lambda_{m+1,n}\\ +\Delta^{m-1}\lambda_{m+1,n+1}\end{pmatrix} \left((m+n)!\,|X_{mn} - 0|\right)^{\frac{1}{m}+n}, \overline{0}\right) = 0.$$

This implies that

$$lim_{mn}|\mu_{mn}(X) - 0| = lim_{mn}\; \overline{d}\left(\left\|\left(\frac{1}{\varphi_{rs}}\sum_{m\in I_{rs}}\sum_{n\in I_{rs}}\begin{pmatrix}\Delta^{m-1}\lambda_{m,n}\\ -\Delta^{m-1}\lambda_{m,n+1}\\ -\Delta^{m-1}\lambda_{m+1,n}\\ +\Delta^{m-1}a_{m+1,n+1}\end{pmatrix} \left((m+n)!\,|X_{mn} - 0|\right)^{1/m+n}\right), \overline{0}\right\|\right)$$

=0.

which yields that $lim_{uv}\mu_{mn}(x) = 0$ and hence $X = (X_{mn}) \in w^2$ is $\lambda -$ convergent to 0.

Let $I^2 -$ be an admissible ideal of $2^{\mathbb{N}\times\mathbb{N}}, \theta_{rs}$ be a double lacunary sequence, $f = (f_{mn})$ be a sequence of Musielak-Orlicz function and $\left(X, \left\|\left(d(x_1, 0), d(x_2, 0), \cdots, d(x_{n-1}, 0)\right)\right\|_p\right)$ be a $p -$ metric space, $q = (q_{mn})$ be double analytic sequence of sequence of positive real numbers. By $w^2(p - X)$ we denote the space of all sequences defined over $\left(X, \left\|\left(d(x_1, 0), d(x_2, 0), \cdots, d(x_{n-1}, 0)\right)\right\|_p\right)$. The following inequality will be used throughout the paper. If $0 \le q_{mn} \le sup q_{mn} = H, K = max(1, 2^{H-1})$ then

$$|a_{mn} + b_{mn}|^{q_{mn}} \le K\{|a_{mn}|^{q_{mn}} + |b_{mn}|^{q_{mn}}\} \quad (2.1)$$

for all m, n and $a_{mn}, b_{mn} \in \mathbb{C}$. Also $|a|^{q_{mn}} \le max(1, |a|^H)$ for all $a \in \mathbb{C}$.

3. Acceleration Convergence of Multiple Sequences of Fuzzy Numbers

In this section some more definitions related to double sequences of fuzzy numbers have been dened and some interesting theorems regarding acceleration convergence of double sequences of fuzzy numbers have been discussed.

3.1. Definition

Let $x = (x_{mn})$ and $y = (y_{mn})$ be two double sequences of real numbers. Then the sequence x is said to converge $P - \chi^2$ faster than the sequence y, written as $x <^P y$, if

$$P - lim_{mn}\left|\frac{\left((m+n)!x_{mn}\right)^{1/m+n}}{\left((m+n)!y_{mn}\right)^{1/m+n}}\right| = 0$$

3.2. Definition

The sequence $x = (x_{mn})$ is said to converge at the same rate in Pringsheim's sense as the sequence $y = (y_{mn})$, written as $x \approx^{P-\chi^2} y$, if

$$0 < P - lim - inf\left|\frac{\left((m+n)!x_{mn}\right)^{1/m+n}}{\left((m+n)!y_{mn}\right)^{1/m+n}}\right| \le P - lim - inf\left|\frac{\left((m+n)!x_{mn}\right)^{1/m+n}}{\left((m+n)!y_{mn}\right)^{1/m+n}}\right| < \infty.$$

3.3. Definition

The four dimensional matrix $A = \left(a_{k,\ell,m,n}\right)$ is said $P -$ accelerate the convergence of the sequence $x = (x_{mn})$ if $Ax <^P x$.

We define the $P -$ acceleration field of A as the set

$$\{x = (x_{mn}) \in w^2 : Ax <^P x\}.$$

Now we define the acceleration convergence of double sequences of fuzzy numbers as follows:

3.4. Definition

Let $X = (X_{mn})$ and $Y = (Y_{mn})$ be two double sequences of fuzzy numbers with $X_{mn} \to \overline{0}$ and $Y_{mn} \to \overline{0}$. Then the sequence X converges to $\overline{0}, P - \chi^2$ faster than the sequence Y converges to $\overline{0}$, written as $X <^{P-\chi^2} Y$, if

$$P - lim_{mn} \left| \frac{\overline{d}\left(\left((m+n)! X_{mn} \right)^{1/m+n}, \overline{0} \right)}{\overline{d}\left(\left((m+n)! Y_{mn} \right)^{1/m+n}, \overline{0} \right)} \right| = 0, \quad \text{provided}$$

$\overline{d}\left(\left((m+n)!\, Y_{mn} \right)^{1/m+n}, \overline{0} \right) \neq 0$ for all $m, n \in \mathbb{N}$.

3.5. Definition

The double sequence $X = (X_{mn})$ converges to $\overline{0}$ at the same rate in Pringsheim's sense as the sequence $Y = (Y_{mn})$ converges to $\overline{0}$, written as $X \approx^{P-\chi^2} Y$, if $0 < P - lim - inf \left| \frac{\overline{d}\left(\left((m+n)! X_{mn} \right)^{1/m+n}, \overline{0} \right)}{\overline{d}\left(\left((m+n)! Y_{mn} \right)^{1/m+n}, \overline{0} \right)} \right| \leq P - lim - inf \left| \frac{\overline{d}\left(\left((m+n)! x_{mn} \right)^{1/m+n}, \overline{0} \right)}{\overline{d}\left(\left((m+n)! y_{mn} \right)^{1/m+n}, \overline{0} \right)} \right| < \infty.$

3.6. Definition

The four dimensional matrix $A = \left(a_{k,\ell,m,n} \right)$ is said $P-$ accelerate the convergence of the sequence $X = (X_{mn})$ if $AX <^P X$.

We define the $P-$ acceleration field of A as the set

$$\{X = (X_{mn}) \in w^2 : AX <^P X\}.$$

3.7. Definition

A matrix transformation associated with the four-dimensional matrix A is said to be an $\chi^{2F} - \chi^{2F}$ if AX is in the set χ^{2F}, whenever X is in χ^{2F} and is anlytic. In the present paper we define the following sequence spaces:

$$\left[\chi^{2q}_{f\mu}, \left\| \left(d(X_1, 0), d(X_2, 0), \cdots, d(X_{n-1}, 0) \right) \right\|^{\varphi}_{p} \right]^{I^2}_{\theta_{rs}}$$

$$= \left\{ r, s \in I_{rs} : \left[f_{mn} \left(\left\| \begin{pmatrix} \mu_{mn}(X), \\ d(X_1, 0), \\ d(X_2, 0), \\ \cdots, \\ d(X_{n-1}, 0) \end{pmatrix} \right\|_p \right) \right]^{q_{mn}} \geq e \right\} \in I^2,$$

$$\left[\Lambda^{2q}_{f\mu}, \left\| \left(d(X_1, 0), d(X_2, 0), \cdots, d(X_{n-1}, 0) \right) \right\|^{\varphi}_{p} \right]^{I^2}_{\theta_{rs}}$$

$$= \left\{ r, s \in I_{rs} : \left[f_{mn} \left(\left\| \begin{pmatrix} \mu_{mn}(X), \\ d(X_1, 0), \\ d(X_2, 0), \\ \cdots, \\ d(X_{n-1}, 0) \end{pmatrix} \right\|_p \right) \right]^{q_{mn}} \geq K \right\} \in I^2,$$

If we take $f_{mn}(X) = X$, we get

$$\left[\chi^{2q}_{f\mu}, \left\| \left(d(X_1, 0), d(X_2, 0), \cdots, d(X_{n-1}, 0) \right) \right\|^{\varphi}_{p} \right]^{I^2}_{\theta_{rs}}$$

$$= \left\{ r, s \in I_{rs} : \left[\left(\left\| \begin{pmatrix} \mu_{mn}(X), \\ d(X_1, 0), \\ d(X_2, 0), \\ \cdots, \\ d(X_{n-1}, 0) \end{pmatrix} \right\|_p \right) \right]^{q_{mn}} \geq \epsilon \right\} \in I^2,$$

$$\left[\Lambda^{2q}_{f\mu}, \left\| \left(d(X_1, 0), d(X_2, 0), \cdots, d(X_{n-1}, 0) \right) \right\|^{\varphi}_{p} \right]^{I^2}_{\theta_{rs}}$$

$$= \left\{ r, s \in I_{rs} : \left[\left(\left\| \mu_{mn}(X), \begin{pmatrix} d(X_1, 0), \\ d(X_2, 0), \\ \cdots, \\ d(X_{n-1}, 0) \end{pmatrix} \right\|_p \right) \right]^{q_{mn}} \geq K \right\} \in I^2$$

If we take $q = (q_{mn}) = 1$, we get

$$\left[\chi^{2}_{f\mu}, \left\| \left(d(X_1, 0), d(X_2, 0), \cdots, d(X_{n-1}, 0) \right) \right\|^{\varphi}_{p} \right]^{I}_{\theta_{rs}}$$

$$= \left\{ r, s \in I_{rs} : \left[f_{mn} \left(\left\| \begin{pmatrix} \mu_{mn}(X), \\ d(X_1, 0), \\ d(X_2, 0), \\ \cdots, \\ d(X_{n-1}, 0) \end{pmatrix} \right\|_p \right) \right] \geq \epsilon \right\} \in I^2,$$

$$\left[\Lambda^{2}_{f\mu}, \left\| \left(d(X_1, 0), d(X_2, 0), \cdots, d(X_{n-1}, 0) \right) \right\|^{\varphi}_{p} \right]^{I^2}_{\theta_{rs}}$$

$$= \left\{ r,s \in I_{rs} : \left[f_{mn} \left(\left\| \mu_{mn}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix} \right\|_p \right) \right] \geq K \right\} \in I^2$$

In the present paper we plan to study some topological properties and inclusion relation between the above defined sequence spaces. $\left[\chi^{2q}_{f\mu}, \left\|\left(d(X_1), d(X_2), \cdots, d(X_{n-1})\right) t \right\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}}$ and $\left[\Lambda^{2q}_{f\mu}, \left\|\left(d(X_1), d(X_2), \cdots, d(X_{n-1})\right)\right\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}}$ which we shall discuss in this paper.

4. Main Results

4.1. Theorem

Let $f = (f_{mn})$ be a acceleration sequence of Musielak-Orlicz function, $q = (q_{mn})$ be a double analytic acceleration sequence of positive real numbers, the acceleration sequence spaces

$\left[\chi^{2q}_{f\mu}, \left\|\left(d(X_1,0), d(X_2,0), \cdots, d(X_{n-1},0)\right)\right\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}}$ and

$\left[\Lambda^{2q}_{f\mu}, \left\|\left(d(X_1,0), d(X_2,0), \cdots, d(X_{n-1},0)\right)\right\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}}$

are linear spaces.

Proof: It is routine verification. Therefore the proof is omitted.

4.2. Theorem

Let $f = (f_{mn})$ be a acceleration sequence of Musielak-Orlicz function, $q = (q_{mn})$ be a double analytic acceleration sequence of positive real numbers, the acceleration sequence space $\left[\chi^{2q}_{f\mu}, \left\|\left(d(X_1,0), d(X_2,0), \cdots, d(X_{n-1},0)\right)\right\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}}$ is a paranormed space with respect to the paranorm defined by $g(x) = inf$

$$\left\{ \left[f_{mn} \left(\left\| \mu_{mn}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix} \right\|_p \right) \right]^{q_{mn}} \leq 1 \right\}$$

Proof: Clearly $g(X) \geq 0$ for $X = (X_{mn}) \in \left[\chi^{2q}_{f\mu}, \left\|\left(d(X_1,0), d(X_2,0), \cdots, d(X_{n-1},0)\right)\right\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}}$

Since $f_{mn}(0) = 0$, we get $g(0) = 0$.

Conversely, suppose that $g(X) = 0$, then

$$inf \left\{ \left[f_{mn} \left(\left\| \mu_{mn}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix} \right\|_p \right) \right]^{q_{mn}} \leq 1 \right\}$$

Suppose that $\mu_{mn}(X) \neq 0$ for each $m,n \in \mathbb{N}$. Then,

$\left\|\mu_{mn}(X), \left(d(X_1,0), d(X_2,0), \cdots, d(X_{n-1},0)\right)\right\|^{\varphi}_{p} \to \infty$. It follows that

$$\left(\left[f_{mn} \left(\left\| \mu_{mn}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix} \right\|_p \right) \right]^{q_{mn}} \right)^{1/H} \to \infty$$

which is a contradiction. Therefore $\mu_{mn}(X) = 0$. Let

$$\left(\left[f_{mn} \left(\left\| \mu_{mn}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix} \right\|_p \right) \right]^{q_{mn}} \right)^{1/H} \leq 1$$

and

$$\left(\left[f_{mn} \left(\left\| \mu_{mn}(Y), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix} \right\|_p \right) \right]^{q_{mn}} \right)^{1/H} \leq 1$$

Then by using Minkowski's inequality, we have

$$\left(\left[f_{mn} \left(\left\| \mu_{mn}(X+Y), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix} \right\|_p \right) \right]^{q_{mn}} \right)^{1/H}$$

$$\leq \left(\left[f_{mn} \left(\left\| \mu_{mn}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix} \right\|_p \right) \right]^{q_{mn}} \right)^{1/H}$$

.

$$+\left(\left[f_{mn}\left(\left\|\mu_{mn}(Y),\begin{pmatrix} d(X_1,0),\\ d(X_2,0),\\ \cdots,\\ d(X_{n-1},0)\end{pmatrix}\right\|_p\right)\right]^{q_{mn}}\right)^{1/H}.$$

So we have

$$g(X+Y)=$$

$$inf\left\{\left[f_{mn}\left(\left\|\mu_{mn}(X+Y),\begin{pmatrix} d(X_1,0),\\ d(X_2,0),\\ \cdots,\\ d(X_{n-1},0)\end{pmatrix}\right\|_p\right)\right]^{q_{mn}}\leq 1\right\}$$

$$\leq inf\left\{\left[f_{mn}\left(\left\|\mu_{mn}(X),\begin{pmatrix} d(X_1,0),\\ d(X_2,0),\\ \cdots,\\ d(X_{n-1},0)\end{pmatrix}\right\|_p\right)\right]^{q_{mn}}\leq 1\right\}$$

$$+inf\left\{\left[f_{mn}\left(\left\|\mu_{mn}(Y),\begin{pmatrix} d(X_1,0),\\ d(X_2,0),\\ \cdots,\\ d(X_{n-1},0)\end{pmatrix}\right\|_p\right)\right]^{q_{mn}}\leq 1\right\}$$

Therefore, $g(X+Y)\leq g(X)+g(Y)$.

Finally, to prove that the scalar multiplication is continuous. Let λ be any complex number. By definition,

$$g(\lambda X)=$$

$$inf\left\{\left[f_{mn}\left(\left\|\mu_{mn}(\lambda X),\begin{pmatrix} d(X_1,0),\\ d(X_2,0),\\ \cdots,\\ d(X_{n-1},0)\end{pmatrix}\right\|_p\right)\right]^{q_{mn}}\leq 1\right\}.$$

Then, $g(\lambda X)=$

$$inf\left\{\begin{matrix}((|\lambda|t)^{\frac{q_{mn}}{H}}:\\ \left[f_{mn}\left(\left\|\mu_{mn}(\lambda X),\begin{pmatrix} d(X_1,0),\\ d(X_2,0),\\ \cdots,\\ d(X_{n-1},0)\end{pmatrix}\right\|_p\right)\right]^{q_{mn}}\leq 1\end{matrix}\right\}.$$

where $t=\frac{1}{|\lambda|}$. Since $|\lambda|^{q_{mn}}\leq max(1,|\lambda|^{sup\, p_{mn}})$, we have

$$g(\lambda X)\leq max(1,|\lambda|^{sup\, p_{mn}})inf$$

$$\left\{t^{q_{mn}/H}:\left[f_{mn}\left(\left\|\mu_{mn}(\lambda X),\begin{pmatrix} d(X_1,0),\\ d(X_2,0),\\ \cdots,\\ d(X_{n-1},0)\end{pmatrix}\right\|_p\right)\right]^{q_{mn}}\leq 1\right\}$$

This completes the proof.

4.3. Proposition

If $0<q_{mn}<p_{mn}<\infty$ for each m and m, then acceleration sequence space

$$\left[\Lambda^{2q}_{f\mu},\left\|\mu_{mn}(X),\begin{pmatrix} d(X_1,0),\\ d(X_2,0),\\ \cdots,\\ d(X_{n-1},0)\end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}\subseteq$$

$$\left[\Lambda^{2p}_{f\mu},\|\mu_{mn}t(X),\begin{pmatrix} d(X_1,0),\\ d(X_2,0),\\ \cdots,\\ d(X_{n-1},0)\end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}$$

Proof: The proof is standard, so we omit it.

4.4. Proposition

(i) If $0<inf\, q_{mn}\leq q_{mn}<1$ then acceleration sequence space

$$\left[\Lambda^{2q}_{f\mu},\left\|\mu_{mn}(X),\begin{pmatrix} d(X_1,0),\\ d(X_2,0),\\ \cdots,\\ d(X_{n-1},0)\end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}$$

$$\subset\left[\Lambda^{2}_{f\mu},\left\|\mu_{mn}(X),\begin{pmatrix} d(X_1,0),\\ d(X_2,0),\\ \cdots,\\ d(X_{n-1},0)\end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}.$$

(ii) If $1\leq q_{mn}\leq sup\, q_{mn}<\infty$, then,

$$\left[\Lambda^{2}_{f\mu},\left\|\mu_{mn}(X),\begin{pmatrix} d(X_1,0),\\ d(X_2,0),\\ \cdots,\\ d(X_{n-1},0)\end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}$$

$$\subset\left[\Lambda^{2q}_{f\mu},\left\|\mu_{mn}(X),\begin{pmatrix} d(X_1,0),\\ d(X_2,0),\\ \cdots,\\ d(X_{n-1},0)\end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}$$

Proof: The proof is standard, so we omit it.

4.5. Proposition

Let $f'=(f'_{mn})$ and $f''=(f''_{mn})$ are acceleration sequences of Musielak-Orlicz functions, we have

$$\left[\Lambda^{2q}_{f'\mu}, \left\|\mu_{mn}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}$$

$$\bigcap \left[\Lambda^{2q}_{f''\mu}, \left\|\mu_{mn}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}$$

$$\subseteq \left[\Lambda^{2q}_{f'+f''\mu}, \left\|\mu_{mn}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}$$

Proof: The proof is easy so we omit it.

4.6. Proposition

For any acceleration sequence of Musielak-Orlicz functions $f = (f_{mn})$ and $q = (q_{mn})$ be double analytic acceleration sequence of positive real numbers.

$$Then \left[\chi^{2q}_{f\mu}, \left\|\mu_{mn}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}$$

$$\subset \left[\Lambda^{2q}_{f\mu}, \left\|\mu_{mn}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}$$

Proof: The proof is easy so we omit it.

4.7. Proposition

The acceleration sequence space

$$\left[\Lambda^{2q}_{f\mu}, \left\|\mu_{mn}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}} \text{ is solid}$$

Proof:

Let $X = (X_{mn}) \in$

$$\left[\Lambda^{2q}_{f\mu}, \left\|\mu_{mn}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}, \text{ (i.e)}$$

$$sup_{mn}\left[\Lambda^{2q}_{f\mu}, \left\|\mu_{mn}(X), \begin{pmatrix} d(X_1,0) \\ ,d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}} < \infty.$$

Let (α_{mn}) be double sequence of scalars such that $|\alpha_{mn}| \leq 1$ for all $m,n \in \mathbb{N} \times \mathbb{N}$. Then we get

$$sup_{mn}\left[\Lambda^{2q}_{f\mu}, \left\|\mu_{mn}(\alpha X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}$$

$$\leq \sup_{mn}\left[\Lambda^{2q}_{f\mu}, \left\|\mu_{mn}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}.$$

This completes the proof.

4.8. Proposition

The sequence space

$$\left[\Lambda^{2q}_{f\mu}, \left\|\mu_{mn}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}} \text{ is monotone}$$

Proof: The proof follows from Proposition 4.9.

4.9. Proposition

If $f = (f_{mn})$ be any acceleration sequence of Orlicz function. Then

$$\left[\Lambda^{2q}_{f\mu}, \left\|\mu_{mn}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|_p^{\varphi^*}\right]^{I^2}_{\theta_{rs}} \subset$$

$$\left[\Lambda^{2q}_{f\mu}, \left\|\mu_{mn}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|_p^{\varphi^{**}}\right]^{I^2}_{\theta_{rs}}$$

if and only if

$$sup_{r,s\geq 1}\frac{\varphi^*_{rs}}{\varphi^{**}_{rs}} < \infty.$$

Proof:

Let $x \in \left[\Lambda^{2q}_{f\mu}, \left\| \mu_{mn}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix} \right\|_p^{\varphi^*}\right]^{I^2}_{\theta_{rs}}$ and

$N = sup_{r,s\geq 1} rac\varphi^*_{rs}\varphi^{**}_{rs} < \infty$. Then we get

$$\left[\Lambda^{2q}_{f\mu}, \left\| \mu_{mn}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix} \right\|_p^{\varphi^{**}_{rs}}\right]^{I^2}_{\theta_{rs}}$$

$$= N\left[\Lambda^{2q}_{f\mu}, \| \mu_{mn} left(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(x_{n-1},0) \end{pmatrix} \right\|_p^{\varphi^*_{rs}}\right]^{I^2}_{\theta_{rs}} = 0.$$

Thus $x \in \left[\Lambda^{2q}_{f\mu}, \left\| \mu_{mn}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix} \right\|_p^{\varphi^{**}}\right]^{I^2}_{\theta_{rs}}$.

Conversely, suppose that

$$\left[\Lambda^{2q}_{f\mu}, \left\| \mu_{mn}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix} \right\|_p^{\varphi^*}\right]^{I}_{N_\theta}$$

$$\subset \left[\Lambda^{2qu}_{f\mu}, \left\| \mu_{mn}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix} \right\|_p^{\varphi^{**}}\right]^{I^2}_{\theta_{rs}}$$ and

$$X \in \left[\Lambda^{2q}_{f\mu}, \left\| \mu_{mn}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix} \right\|_p^{\varphi^*}\right]^{I^2}_{\theta_{rs}}$$ Then,

$$\left[\Lambda^{2q}_{f\mu}, \left\| \mu_{mn}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix} \right\|_p^{\varphi^*}\right]^{I^2}_{\theta_{rs}} < e,$$

for every $e > 0$. Suppose that $sup_{r\geq 1,s\geq 1} \frac{\varphi^*_{rs}}{\varphi^{**}_{rs}} = \infty$, then there exists a acceleration sequence of members $(r_{jk}\, s_{jk})$ such that $lim_{r_{jk}\, s_{jk} \to \infty} \frac{\varphi^*_{r_{jk}\, s_{jk}}}{\varphi^{**}_{r_{jk}\, s_{jk}}} = \infty$. Hence, we have

$$\left[\Lambda^{2qu}_{f\mu}, \left\| \mu_{mn}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix} \right\|_p^{\varphi^*_{rs}}\right]^{I^2}_{\theta_{rs}} = \infty.$$

Therefore

$$X \notin \left[\Lambda^{2q}_{f\mu}, \left\| \mu_{mn}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix} \right\|_p^{\varphi^{**}}\right]^{I^2}_{\theta_{rs}},$$ which

is a contradiction. This completes the proof.

4.10. Proposition

If $f = (f_{mn})$ be any acceleration of Musielak Orlicz function. Then

$$\left[\Lambda^{2q}_{f\mu}, \left\| \mu_{mn}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix} \right\|_p^{\varphi^*}\right]^{I^2}_{\theta_{rs}}$$

$$= \left[\Lambda^{2q}_{f\mu}, \left\| \mu_{mn}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix} \right\|_p^{\varphi^{**}}\right]^{I^2}_{\theta_{rs}}$$ if and only if

$sup_{r,s\geq 1} \frac{\varphi^*_{rs}}{\varphi^{**}_{rs}} < \infty, sup_{r,s\geq 1} rac\varphi^{**}_{rs}\varphi^*_{rs} > \infty.$

Proof: It is easy to prove so we omit.

4.11. Proposition

The acceleration of sequence space

$\left[\chi^{2q}_{f\mu}, \| \mu_{mn}(X), (d(X_1,0), d(X_2,0), \cdots, d(X_{n-1},0)) \|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}$

is not solid

Proof: The result follows from the following example.

Example: Consider

$$X = (X_{mn}) = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ . & & & \\ . & & & \\ . & & & \\ 1 & 1 & \dots & 1 \end{pmatrix} \in$$

$$\left[\chi^{2q}_{f\mu}, \left\|\mu_{mn}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}}.$$

Let $\alpha_{mn} =$

$$\begin{pmatrix} -1^{m+n} & -1^{m+n} & \dots & -1^{m+n} \\ -1^{m+n} & -1^{m+n} & \dots & -1^{m+n} \\ . & & & \\ . & & & \\ . & & & \\ -1^{m+n} & -1^{m+n} & \dots & -1^{m+n} \end{pmatrix}, for\ all \text{ m,n} \in \mathbb{N}.$$

Then $\alpha_{mn} X_{mn} \notin$

$$\left[\chi^{2q}_{f\mu}, \left\|\mu_{mn}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}}.$$

Hence

$$\left[\chi^{2q}_{f\mu}, \left\|\mu_{mn}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}}$$ is not solid.

4.12. Proposition

The acceleration sequence space

$$\left[\chi^{2q}_{f\mu}, \left\|\mu_{mn}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}}$$ is not monotone.

Proof: The proof follows from Proposition 4.13.

4.13. Proposition

The acceleration sequence spaces

$$\left[\chi^{2q}_{f\mu}, \left\|\mu_{mn}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}}$$ and

$$\left[\chi^{2q}_{f\mu}, \left\|\mu_{mn}(Y), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}}$$ be two

elements ${}_2S^{BF}_0$ such that

$$\left[\chi^{2q}_{f\mu}, \left\|\mu_{mn}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}} < P - \chi^2$$

$$\left[\chi^{2q}_{f\mu}, \left\|\mu_{mn}(Y), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}}$$, then there exists an

element

$$\left[\chi^{2q}_{f\mu}, \left\|\mu_{mn}(Z), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}} \in_2 S^{BF}_0$$ such that

$$\left[\chi^{2q}_{f\mu}, \left\|\mu_{mn}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}} < P - \chi^2$$

$$\left[\chi^{2q}_{f\mu}, \left\|\mu_{mn}(Z), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}} < P - \chi^2$$

$$\left[\chi^{2q}_{f\mu}, \left\|\mu_{mn}(Y), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}}.$$

Proof: Let $\left[\chi^{2q}_{f\mu}, \left\|\mu_{mn}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}}$,

$$\left[\chi_{f\mu}^{2q}, \left\|\mu_{mn}(Y), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|_p^{\varphi}\right]_{\theta_{rs}}^{I^2} \in_2 S_0^{BF}$$

be such that

$$\left[\chi_{f\mu}^{2q}, \left\|\mu_{mn}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|_p^{\varphi}\right]_{\theta_{rs}}^{I^2} < P - \chi^2$$

$$\left[\chi_{f\mu}^{2q}, \left\|\mu_{mn}(Y), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|_p^{\varphi}\right]_{\theta_{rs}}^{I^2}.$$

Define the sequence

$$\left[\chi_{f\mu}^{2q}, \left\|\mu_{mn}(Z), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|_p^{\varphi}\right]_{\theta_{rs}}^{I^2} < P - \chi^2$$

$$\left[\chi_{f\mu}^{2q}, \left\|\mu_{mn}(Z), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0)) \end{pmatrix}\right\|_p^{\varphi}\right]_{\theta_{rs}}^{I^2}$$

as follows:

$$\left[\chi_{f\mu}^{2q}, \left\|\mu_{mn}(Z), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|_p^{\varphi}\right]_{\theta_{rs}}^{I^2} =$$

$$\left[\chi_{f\mu}^{2q}, \left\|\mu_{mn}\left(X^{1/5}\right), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|_p^{\varphi}\right]_{\theta_{rs}}^{I^2} \otimes$$

$$\left[\chi_{f\mu}^{2q}, \left\|\mu_{mn}\left(Y^{4/5}\right), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|_p^{\varphi}\right]_{\theta_{rs}}^{I^2}$$

This implies that

$$\left[\chi_{f\mu}^{2q}, \left\|\mu_{mn}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|_p^{\varphi}\right]_{\theta_{rs}}^{I^2} <^{P-\chi^2}$$

$$\left[\chi_{f\mu}^{2q}, \left\|\mu_{mn}(Z), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|_p^{\varphi}\right]_{\theta_{rs}}^{I^2} <^{P-\chi^2}$$

$$\left[\chi_{f\mu}^{2q}, \left\|\mu_{mn}(Y), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|_p^{\varphi}\right]_{\theta_{rs}}^{I^2}.$$

4.14. Theorem

Let $\left[\chi_{f\mu}^{2q}, \left\|\mu_{mn}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|_p^{\varphi}\right]_{\theta_{rs}}^{I^2} <^{P-\chi^2}$

$$\left[\chi_{f\mu}^{2q}, \left\|\mu_{mn}(Y), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|_p^{\varphi}\right]_{\theta_{rs}}^{I^2}$$

and

$$\left[\chi_{f\mu}^{2q}, \left\|\mu_{mn}(Y), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|_p^{\varphi}\right]_{\theta_{rs}}^{I^2} <^{P-\chi^2}$$

$$\left[\chi_{f\mu}^{2q}, \left\|\mu_{mn}(Z), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|_p^{\varphi}\right]_{\theta_{rs}}^{I^2}, \text{ then}$$

$$\left[\chi_{f\mu}^{2q}, \left\|\mu_{mn}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|_p^{\varphi}\right]_{\theta_{rs}}^{I^2} <^{P-\chi^2}$$

$$\left[\chi_{f\mu}^{2q}, \left\|\mu_{mn}(Z), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|_p^{\varphi}\right]_{\theta_{rs}}^{I^2}.$$

Proof: The proof is omitted as it is straight forward.

4.15. Theorem

Let A be a non-negative $\chi^{2BF} - \chi^{2BF}$ summability matrix and let

$$\left[\chi^{2q}_{f\mu}, \left\| \mu_{mn}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix} \right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}$$ and

$$\left[\chi^{2q}_{f\mu}, \left\| \mu_{mn}(Y), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix} \right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}$$ be two

elements in ${}_2\ell^F$ such that

$$\left[\chi^{2q}_{f\mu}, \left\| \mu_{mn}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix} \right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}} <^{P-\chi^2},$$

$$\left[\chi^{2q}_{f\mu}, \left\| \mu_{mn}(Y), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix} \right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}$$ with

$$\left[\chi^{2q}_{f\mu}, \left\| \mu_{mn}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix} \right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}} \in_2 S_0^{BF}$$ and

$$\left[\chi^{2q}_{f\mu}, \left\| \mu_{mn}(Y), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix} \right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}} \in_2 S_0^{F}$$ for

some $\delta > 0$, then

$$\left[\chi^{2q}_{f\mu}, \left\| \mu_{mn}(AX), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix} \right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}} <^{P-\chi^2}$$

$$\left[\chi^{2q}_{f\mu}, \left\| \mu_{mn}(AY), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix} \right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}$$

Proof: Since

$$\left[\chi^{2q}_{f\mu}, \left\| \mu_{mn}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix} \right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}} <^{P-\chi^2}$$

$$\left[\chi^{2q}_{f\mu}, \left\| \mu_{mn}(Y), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix} \right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}},$$

then there exists a analytic double sequence

$$\left[\chi^{2q}_{f\mu}, \left\| \mu_{mn}(Z), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix} \right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}$$

with Pringsheim's limit zero such that

$$\left[\chi^{2q}_{f\mu}, \left\| \mu_{mn}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix} \right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}} =$$

$$\left[\chi^{2q}_{f\mu}, \left\| \mu_{mn}(Y), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix} \right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}} \otimes$$

$$\left[\chi^{2q}_{f\mu}, \left\| \mu_{mn}(Z), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix} \right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}.$$

For each k and ℓ, we have

$$\frac{\left[\chi^{2q}_{f\mu},\left\|\mu_{mn}(AX),\begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0)\end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}}{\left[\chi^{2q}_{f\mu},\left\|\mu_{mn}(AY),\begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0)\end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}} = \frac{sup_{r,s\geq k,\ell}\left[\chi^{2q}_{f\mu},\left\|\mu_{mn}(AX),\begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0)\end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}}{sup_{r,s\geq k,\ell}\left[\chi^{2q}_{f\mu},\left\|\mu_{mn}(AY),\begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0)\end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}} =$$

$$\frac{sup_{r,s\geq k,\ell}\sum_{m,n=1,1}^{\infty,\infty}\left[\chi^{2q}_{f\mu},\left\|\mu_{mn}(AX),\begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0)\end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}}{sup_{r,s\geq k,\ell}\sum_{m,n=1,1}^{\infty,\infty}\left[\chi^{2q}_{f\mu},\left\|\mu_{mn}(AY),\begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0)\end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}} = \frac{sup_{r,s\geq k,\ell}\sum_{m,n=1,1}^{\infty,\infty}\left[\chi^{2q}_{f\mu},\left\|\mu_{mn}\left(A(Y\otimes Z)\right),\begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0)\end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}}{sup_{r,s\geq k,\ell}\sum_{m,n=1,1}^{\infty,\infty}\left[\chi^{2q}_{f\mu},\left\|\mu_{mn}(AY),\begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0)\end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}}$$

$$= \frac{sup_{r,s\geq k,\ell}\sum_{m,n=1,1}^{\infty,\infty} a_{r,s,m,n}\left[\chi^{2q}_{f\mu},\left\|\mu_{mn}(Y),\begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0)\end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}\left[\chi^{2q}_{f\mu},\left\|\mu_{mn}(Z),\begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0)\end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}}{\delta sup_{r,s\geq k,\ell}\sum_{m,n=1,1}^{\infty,\infty} a_{r,s,m,n}}$$

Since Y and Z are double analytic sequences with Z is in χ^{2F} and A is a non-negative $\chi^{2F}-\chi^{2F}$ matrix, then

$$P - sup_{r,s\geq k,\ell}\sum_{m,n=1,1}^{\infty,\infty} a_{r,s,m,n}$$

$$\left[\chi^{2q}_{f\mu},\left\|\mu_{mn}(Y),\begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0)\end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}$$

$$\left[\chi^{2q}_{f\mu},\left\|\mu_{mn}(Z),\begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0)\end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}} = 0$$

Hence

$$P - lim\frac{\left[\chi^{2q}_{f\mu},\left\|\mu_{mn}(AX),\begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0)\end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}}{\left[\chi^{2q}_{f\mu},\left\|\mu_{mn}(AY),\begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0)\end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}} \leq 0 \quad (4.1)$$

In a similar manner we can establish

$$P - lim\frac{\left[\chi^{2q}_{f\mu},\left\|\mu_{mn}(AX),\begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0)\end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}}{\left[\chi^{2q}_{f\mu},\left\|\mu_{mn}\left(AYright),\begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0)\end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}} \geq 0 \quad (4.2)$$

From (4.1) and (4.2), we have

$$P - lim\frac{\left[\chi^{2q}_{f\mu},\left\|\mu_{mn}(AX),\begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0)\end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}}{\left[\chi^{2q}_{f\mu},\left\|\mu_{mn}(AY),\begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0)\end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}} = 0$$

which implies

$$\left[\chi^{2q}_{f\mu}, \left\|\mu_{mn}(AY), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}} <^{P-\chi^2}$$

$$\left[\chi^{2q}_{f\mu}, \left\|\mu_{mn}(AY), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}}.$$

4.16. Theorem

Let the acceleration sequence of $\left[\chi^{2q}_{f\mu}, \left\|\mu_{mn}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}} \in S_0^{2BF}$ and A be a transformation such that

$$\left[\chi^{2q}_{f\mu}, \left\|\mu_{mn}(AX), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}} <^{P-\chi^2}.$$

$\left[\chi^{2q}_{f\mu}, \left\|\mu_{mn}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}}$. Then there exists a acceleration sequence of $\left[x^{2q}_{f\mu}, \left\|\mu_{mn}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}} \in \chi^{2BF}$ such that

$$\left[x^{2q}_{f\mu}, \left\|\mu_{mn}(Y), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}} =$$

$\left[\chi^{2q}_{f\mu}, \left\|\mu_{mn}(Y), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}}$ a.a (m,n) and

$$\left[\chi^{2q}_{f\mu}, \left\|\mu_{mn}(AY), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}} <^{P-\chi^2}$$

$$\left[\chi^{2q}_{f\mu}, \left\|\mu_{mn}(Y), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}}$$

Proof: Let the acceleration sequence $\left[\chi^{2q}_{f\mu}, \left\|\mu_{mn}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}} \in S_0^{2BF}$ Then there exists a subset $B_1 \subset \mathbb{N}\times\mathbb{N}$ with $\delta_2(B_1)=1$ such that

$$P - \left[\chi^{2q}_{f\mu}, \left\|\mu_{mn}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}} = 0, \text{ over } B_1.$$

Let the acceleration sequence $\left[\chi^{2q}_{f\mu}, \left\|\mu_{m_k n_\ell}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}} \in S_0^{2BF}$. Then there exists a subset $B_2 \subset \mathbb{N}\times\mathbb{N}$ with $\delta_2(B_2)=1$ such that

$$P - \left[\chi^{2q}_{f\mu}, \left\|\mu_{m_k n_\ell}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}} = 0, \text{ over } B_2.$$

Since

$$\left[\chi^{2q}_{f\mu}, \left\|\mu_{mn}(AX), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}} <^{P-\chi^2}$$

$$\left[\chi^{2q}_{f\mu}, \left\|\mu_{mn}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|^{\varphi}_{p}\right]^{I^2}_{\theta_{rs}},$$ we have

$$P-lim\frac{\left[\chi^{2q}_{f\mu},\left\|\mu_{m_k n_\ell}(X),\begin{pmatrix} d(X_1,0),\\ d(X_2,0),\\ \cdots,\\ d(X_{n-1},0)\end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}}{\left[\chi^{2q}_{f\mu},\left\|\mu_{mn}(X),\begin{pmatrix} d(X_1,0),\\ d(X_2,0),\\ \cdots,\\ d(X_{n-1},0)\end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}}=0.$$

Then there exists a subset $B_3\subset\mathbb{N}\times\mathbb{N}$ with $\delta_2(B_3)=1$ such that

$$P-lim\frac{\left[\chi^{2q}_{f\mu},\left\|\mu_{m_k n_\ell}(X),\begin{pmatrix} d(X_1,0),\\ d(X_2,0),\\ \cdots,\\ d(X_{n-1},0)\end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}}{\left[\chi^{2q}_{f\mu},\left\|\mu_{mn}(X),\begin{pmatrix} d(X_1,0),\\ d(X_2,0),\\ \cdots,\\ d(X_{n-1},0)\end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}}=0,$$

over B_3.

Let $D=B_1\cap B_2\cap B_3$. Then clearly $\delta_2(D)=1$.

For $r\neq m_k, s\neq n_\ell\in\mathbb{N}\times\mathbb{N}$, let us define the acceleration sequence

$$\left[\chi^{2q}_{f\mu},\left\|\mu_{rs}(Y),\begin{pmatrix} d(X_1,0),\\ d(X_2,0),\\ \cdots,\\ d(X_{n-1},0)\end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}=$$

$$\begin{cases}\dfrac{\left[\chi^{2q}_{f\mu},\left\|\mu_{rs}(X),\begin{pmatrix} d(X_1,0),\\ d(X_2,0),\\ \cdots,\\ d(X_{n-1},0)\end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}}{(rs)^{-3}}, & \text{if } (r,s)\in D; \quad and\\ 0, & \text{otherwise.}\end{cases}$$

$$\left[\chi^{2q}_{f\mu},\left\|\mu_{m_k n_\ell}(Y),\begin{pmatrix} d(X_1,0),\\ d(X_2,0),\\ \cdots,\\ d(X_{n-1},0)\end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}=$$

$$\begin{cases}\dfrac{\left[\chi^{2q}_{f\mu},\left\|\mu_{m_k n_\ell}(X),\begin{pmatrix} d(X_1,0),\\ d(X_2,0),\\ \cdots,\\ d(X_{n-1},0)\end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}}{\left[\chi^{2q}_{f\mu},\left\|\mu_{mn}(Y),\begin{pmatrix} d(X_1,0),\\ d(X_2,0),\\ \cdots,\\ d(X_{n-1},0)\end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}(mn)^{-3}}, & \text{if } (k,\ell)\in D;\\ 0, & \text{otherwise.}\end{cases}$$

Then we have

$$\left[\chi^{2q}_{f\mu},\left\|\mu_{mn}(Y),\begin{pmatrix} d(X_1,0),\\ d(X_2,0),\\ \cdots,\\ d(X_{n-1},0)\end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}\in S_0^{2BF}.$$

Then we have

$$\left[\chi^{2q}_{f\mu},\left\|\mu_{mn}(X),\begin{pmatrix} d(X_1,0),\\ d(X_2,0),\\ \cdots,\\ d(X_{n-1},0)\end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}$$

$$=\left[\chi^{2q}_{f\mu},\left\|\mu_{mn}(Y),\begin{pmatrix} d(X_1,0),\\ d(X_2,0),\\ \cdots,\\ d(X_{n-1},0)\end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}$$

a.a.(m,n) and this implies:

$$A\left[\chi^{2q}_{f\mu},\left\|\begin{pmatrix}\mu_{mn}(Y),(d(X_1,0),\\ d(X_2,0),\\ \cdots,\\ d(X_{n-1},0)\end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}<^{P-\chi^2}$$

$$\left[\chi^{2q}_{f\mu},\left\|\mu_{mn}(Y),\begin{pmatrix} d(X_1,0),\\ d(X_2,0),\\ \cdots,\\ d(X_{n-1},0)\end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}.$$

4.17. Theorem

Let the acceleration sequence of

$$\left[\chi^{2q}_{f\mu},\left\|\mu_{mn}(X),\begin{pmatrix} d(X_1,0),\\ d(X_2,0),\\ \cdots,\\ d(X_{n-1},0)\end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}\in S_0^{2BF} \quad \text{and}$$

A be a acceleration subsequence transformation such that

$$\left[\chi^{2q}_{f\mu},\left\|\mu_{mn}(AX),\begin{pmatrix} d(X_1,0),\\ d(X_2,0),\\ \cdots,\\ d(X_{n-1},0)\end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}<^{P-\chi^2}$$

$$\left[\chi^{2q}_{f\mu},\left\|\mu_{mn}(X),\begin{pmatrix} d(X_1,0),\\ d(X_2,0),\\ \cdots,\\ d(X_{n-1},0)\end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}.$$

Then there exists a acceleration sequence of

$$\left[\chi^{2q}_{f\mu}, \left\|\mu_{mn}(Y), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}} \in$$

S_0^{2BF} such that

$$\left[\chi^{2q}_{f\mu}, \left\|\mu_{mn}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}} <^{P-\chi^2}$$

$$\left[\chi^{2q}_{f\mu}, \left\|\mu_{mn}(Y), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}} \text{ and}$$

$$\left[\chi^{2q}_{f\mu}, \left\|\mu_{mn}(AY), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}} <^{P-\chi^2}$$

$$\left[\chi^{2q}_{f\mu}, \left\|\mu_{mn}(Y), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, d \\ (X_{n-1},0) \end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}.$$

Proof: Consider the acceleration sequence

$$\left[\chi^{2q}_{f\mu}, \left\|\mu_{mn}(Y), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}$$

$$= \left[\chi^{2q}_{f\mu}, \left\|\mu_{mn}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}$$

for all $m, n \in \mathbb{N}$. Then clearly

$$\left[\chi^{2q}_{f\mu}, \left\|mu_{mn}(Y), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}} \in S_0^{2BF}$$

such that

$$\left[\chi^{2q}_{f\mu}, \left\|\mu_{mn}(X), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}} <^{P-\chi^2}$$

$$\left[\chi^{2q}_{f\mu}, \left\|\mu_{mn}(Y), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}} \quad and$$

$$\left[\chi^{2q}_{f\mu}, \left\|\mu_{mn}(AY), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}} <^{P-\chi^2}$$

$$\left[\chi^{2q}_{f\mu}, \left\|\mu_{mn}(Y), \begin{pmatrix} d(X_1,0), \\ d(X_2,0), \\ \cdots, \\ d(X_{n-1},0) \end{pmatrix}\right\|_p^{\varphi}\right]^{I^2}_{\theta_{rs}}.$$

5. Conclusion

In this paper all the definitions are newly constructed and then construct with difference of double sequence space of χ^2 of new theorems statement of proves are formulated. But corresponding paper deals with metric condition of double sequence adopt with randomness and acceleration is a new contribution.

6. Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this research paper.

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