Journal of Mathematical Sciences 3 (2016) 90-96
ISSN 2372-5214

Betty Jones & Sisters Publishing
http://www.bettyjonespub.com

The Poisson Summation Formula for Functions of Bounded Variation

Francisco J. Mendoza Torres, Eder Cardoso Garcia

Facultad de Cs. Físico Matemáticas, Benemérita Universidad Autónoma de Puebla, 18 Sur and San Claudio, 72570, Puebla, Pue., Mexico

Abstract: We present the Poisson summation formula for functions of bounded variation which vanish at infinity. This formula provides a relationship between the symmetrical series of a function and its Fourier transform, both of them evaluated in the integer numbers. Our exposure is based on a part of the article [Mathematical Notes, vol.61, no. 2, 1997] of R. Trigub, and it uses the Henstock-Kurzweil integral to develop it.

Keywords: Poisson formula, Fourier transform, bounded variation function, Henstock-Kurzweil integral.

1. Introduction

Given a function $f:\mathbb{R}\to\mathbb{R}$, its Fourier transform at $\omega\in\mathbb{R}$ is defined as

$$\hat{f}(\omega)=\int_{-\infty}^{\infty} f(x)e^{-ix\omega}\,dx, \qquad (1)$$

always that this integral exists. This fact is dependent upon the type of integral employed. It is known that if $f\in L^1(\mathbb{R})$ its Fourier transform is defined for all $s\in\mathbb{R}$. However, even with the integral of Lebesgue, this transform may not exist if the function belongs to other spaces, for example if the function is of bounded variation in $\mathbb{R}$. In this case we must use improper integrals which are not absolute like the Henstock-Kurzweil integral. Given $f\in L^1(\mathbb{R})$, the Poisson summation formula equals, under additional conditions, the symmetrical series with terms $f(n,x)$, and the symmetric series with terms $\hat{f}(n,x)$, $n\in\mathbb{Z}$. In other words, the next equality holds:

$$\sum_{n\in\mathbb{Z}} f(n,x)\Phi(n,x)=\sum_{n\in\mathbb{Z}}\hat{f}(n,x)\Psi(n,x), \qquad (2)$$

where $\Phi(n,x)$ and $\Psi(n,x)$ are terms that are relatively easy to handle.

Corresponding author: Francisco J. Mendoza Torres, Ph.D., research fields: Integral Theory, Fourier Analysis. E-mail: jmendoza@fcfm.buap.mx.

A fundamental problem of the Fourier analysis is the inversion. It consists of regaining the original function. The Poisson identity is related to this process. If we have $\hat{f}$, the formula (2) provides us information about the discrete behavior of the function f. So with the help of other methods we can find an approximation to the function f. This process is widely used in the sciences which need recovering signals. In this case, it is used in the reconstruction of a band-limited signal, that is, when $\hat{f}$ has compact support, see [1-2]. In the mathematical literature it is not easy to find procedures of the Poisson summation formula that require different methods to the Lebesgue integral theory. Therefore it might be interesting to study this formula in a different context.

We present a version of the Poisson's formula for functions of bounded variation which vanish at infinity, $BV_0(\mathbb{R})$. This space is not contained in any $L^p(\mathbb{R})$ when $p\in[1,\infty)$. For example, $f:\mathbb{R}\to\mathbb{R}$ defined by

$$f(x)=\begin{cases}\dfrac{1}{|x|^{\frac{1}{p}}} & \text{if } x\notin[-1,1]\\ 0 & \text{if } x\in[-1,1],\end{cases}$$

belongs to $BV_0(\mathbb{R})\backslash L^p(\mathbb{R})$.

Let $I\subseteq\mathbb{R}$ be a closed interval which may be bounded or unbounded. We denoted by $HK(I)$ the vector space of Henstock-Kurzweil integrable functions on I, and by $BV(I)$ the vector space of functions of bounded variation on I. The space $HK(I)$ is a semi-normed space with the Alexiewicz semi-norm

$$||f||_A = \sup_{[c,d]\subset I}\left|\int_c^d f(t)dt\right|. \quad (3)$$

In [3] is proved that the set $HK(\mathbb{R})\cap BV(\mathbb{R})$ contains enough functions that are not in $L^1(\mathbb{R})$ and that is a dense subset of $BV_0(\mathbb{R})\cap L^2(\mathbb{R})$, with respect to the L^2-norm. Since $L^1(\mathbb{R})\subset HK(\mathbb{R})$, then we have that $L^1(\mathbb{R})\cap BV(\mathbb{R})\subset HK(\mathbb{R})\cap BV(\mathbb{R})\subset BV_0(\mathbb{R})$. Thus, a motivation to expose the Poisson's formula on $BV_0(\mathbb{R})$ is based on the next result.

Theorem 1.1 [2, Exercise 4.2.9] If $f\in L^1(\mathbb{R})\cap BV(\mathbb{R})$ and $f(x)=\frac{1}{2}[f(x+)+f(x-)]$, then

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty}\hat{f}(n).$$

1.1. Main Result. Considering the above, the main result is the following: If $f\in BV_0(\mathbb{R})$ and $2f(k)=f(k+)+f(k-)$ for all $k\in\mathbb{Z}$, then for $x\not\equiv 0\ (mod\ 2\pi)$ it satisfies that

$$\sum_{k=-\infty}^{\infty} f(k)e^{ikx} = \sqrt{2\pi}\sum_{k=-\infty}^{\infty}\hat{f}(2k\pi - x). \quad (4)$$

Note that unlike the versions for $L^1(\mathbb{R})$ functions, the formula (4) does not allow us to evaluate at $x=0$. However, for practical purposes this is not a problem. The proof of the Poisson summation formula in $BV_0(\mathbb{R})$ presented here is based in the work of Roald Trigub in [4]. Our contribution is to expose in detail some aspects that may contribute to the use of the Henstock-Kurzweil integral.

2. Preliminaries

The definition of Henstock-Kurzweil integral is available in [5-6]. In this section we present some basic concepts related to this integral which also are taken from [5-6]. To simplify sometimes we say that a function is HK integrable if it is Henstock-Kurzweil integrable.

2.1. Functions of Bounded Variation

Supposing that the closed interval I is not bounded, then the function f is of bounded variation on I if it is of bounded variation on each compact interval $I_0\subset I$, and there exists $M>0$ which is a uniform bound of the elements $\mathrm{Var}(f;I_0)$. The total variation of f on I is defined as

$$Var(f;I) = sup\left\{\begin{array}{c} Var(f;I_0): \\ I_0 \text{ is a compact interval contained in } I\end{array}\right\}.$$

Some properties of the vector space of bounded variation functions defined on I, $BV(I)$, are the following:

- Jordan decomposition: $f\in BV(I)$ if and only if there exist f_1 and f_2 which are increasing bounded functions such that $f=f_1-f_2$.
- If I is a non bounded interval, then $\lim_{t\to\pm\infty} f(t)$ exist.

A function f defined on the interval I is of locally bounded variation if it is of bounded variation on any compact interval contained in I. The space of those functions is denoted by $BV_{loc}(I)$. We will refer to $BV_0(I)$ as the subspace of functions belonging to $BV(I)$ such that vanishing at $\pm\infty$.

2.2. The Henstock-Kurzweil Integral

The following theorem plays an important role in the HK-integral theory. We expose the case for $[a,\infty]$. The other cases are analogs.

Theorem 2.1 (Hake's Theorem). $f\in HK([a,\infty])$ if and only if, for all c, ε such that $c>a$, $c-a>\varepsilon>0$, it holds that $f\in HK([a+\varepsilon,c])$ and

$$\lim_{\substack{\varepsilon\to 0\\ c\to\infty}}\int_{a+\varepsilon}^{c} f(t)\,dt$$

exist. The limit will be $\int_a^\infty f(t)\,dt$.

The function f defined on the closed interval I is locally Henstock-Kurzweil integrable if it is Henstock-Kurzweil integrable on every compact interval contained in I. The set of locally Henstock-Kurzweil integrable functions on I is denote by $HK_{loc}(I)$. Similarly to the product of two Lebesgue integrable functions, the product of two HK integrable functions is not necessarily HK integrable. The multipliers in $HK(I)$ are the bounded variation functions.

Multiplier Theorem [5]**.** If $f \in HK([a,\infty])$ and $\varphi \in BV([a,\infty])$, then $f\varphi \in HK([a,\infty])$ and

$$\int_a^\infty f\,\varphi = \lim_{b\to\infty}\left[F(b)\varphi(b) - \int_a^b F\,d\varphi\right]. \qquad (5)$$

The integrals on the right side are Riemann-Stieljes integrals. We have analogous results on $[-\infty,b]$ and $[-\infty,\infty]$.

The Chartier-Dirichlet theorem provides conditions for that the product of two functions will be HK integrable. That theorem is the following.

Theorem 2.2. Let $f,\varphi: [a,\infty] \to \mathbb{R}$ and suppose that:

- $f \in HK([a,c])$ for all $c \geq a$ and $F(x) = \int_a^x f$ is bounded on $[a,\infty)$,
- φ is monotone on $[a,\infty]$ and $\lim_{x\to\infty}\varphi(x) = 0$.

Then $f\varphi \in HK([a,\infty])$.

In order to simplify our discussion, we recall the following lemma, which is available in [7].

Lemma 2.3. Let f and α functions defined on $[a,b]$, the firts continuous and the second of bounded variation. Then

$$\left|\int_a^b f\,d\alpha\right| \leq \sup_{t\in[a,b]}|f(t)|\,Var(\alpha;[a,b]).$$

2.3. The Fourier transform in BV_0

At [8] is proved the Riemann-Lebesgue lemma for functions of bounded variation. This lemma is the following.

Lemma 2.4. If $f \in BV_0(\mathbb{R})$, then the Fourier transform $\hat{f}(\omega)$ exists for all $\omega \in \mathbb{R}\setminus\{0\}$, and has the next properties:

- $\hat{f}: \mathbb{R} \to \mathbb{C}$ is continuous in $\mathbb{R}\setminus\{0\}$.
- $\lim_{|\omega|\to\infty}\hat{f}(\omega) = 0$.

In general the Fourier transform for functions in $BV_0(\mathbb{R})$ is not defined in $\omega = 0$. For example, if f is such that $f(t) = \frac{1}{t}$ for $t \in (-\infty,-1)\cup(1,\infty)$ and $f(t) = 0$ for $t \in [-1,1]$, then f belongs to $BV_0(\mathbb{R})$ and its Fourier transform is not defined in 0.

Lemma 2.5. Let $f \in BV_0(\mathbb{R})$ and $\omega \neq 0$, then

$$\hat{f}(\omega) = -\frac{i}{\omega}\int_{-\infty}^{\infty} e^{-iu\omega}\,df(u) \qquad \text{and}$$

$$\left|\hat{f}(\omega)\right| = \left|\int_{-\infty}^{\infty} e^{-i\omega t}\,f(t)dt\right| \leq \frac{V(f,\mathbb{R})}{|\omega|}.$$

Proof. We have that $h(u) = e^{-iu\omega}$ belongs to $HK_{loc}(\mathbb{R})$, then using the Multiplier Theorem on $[a,b]$:

$$\int_a^b f(u)e^{-iu\omega}\,du$$
$$= \frac{i}{\omega}\left\{(e^{-ib\omega} - e^{-ia\omega})f(b) - \int_a^b [e^{-iu\omega} - e^{-ia\omega}]\,df(u)\right\}$$
$$= \frac{i(e^{-ib\omega} - e^{-ia\omega})}{\omega}f(b) - i\int_a^b \frac{e^{-iu\omega}}{\omega}df(u) + \frac{ie^{-ia\omega}}{\omega}\int_a^b d\,f(u)$$
$$= \frac{i(e^{-ib\omega} - e^{-ia\omega})f(b)}{\omega} + \frac{ie^{-ia\omega}}{\omega}[f(b) - f(a)] - \frac{i}{\omega}\int_a^b e^{-iu\omega}\,df(u).$$

Therefore by Lemma 2.3:

$$\left|\int_a^b f(u)e^{-iu\omega}\,du\right|$$
$$\leq \left|\frac{f(b)}{\omega}\right| + \left|\frac{f(b)-f(a)}{\omega}\right| + \left|\frac{1}{\omega}\int_a^b e^{-iu\omega}\,df(u)\right|$$
$$\leq \left|\frac{f(b)}{\omega}\right| + \left|\frac{f(b)-f(a)}{\omega}\right| + \frac{V(f;[a,b])}{|\omega|}$$

Since $\hat{f}(\omega)$ is defined as a Henstock-Kurzweil integral on $(-\infty,\infty)$ and $f \in BV_0(\mathbb{R})$, then, as $a \to -\infty$ and $b \to \infty$, we get the proof of this lemma. □

Remark 1. Let $x_0 \in \mathbb{R}$ fixed. We know that for each $n \in \mathbb{N}$, see [1, p. 175]

$$1 + 2\cos x_0 + 2\cos 2x_0 + 2\cos 3x_0 + \cdots + 2\cos nx_0 = \frac{\sin(n+1/2)x_0}{\sin(x_0/2)}$$

Therefore

$$\left|\sum_{k=-n}^{n} e^{ikx_0}\right| \le \left|\sum_{k=-n}^{n} \cos kx_0\right| + \left|i\sum_{k=-n}^{n} \sin kx_0]\right| =$$

$$\left|\sum_{k=-n}^{n} \cos kx_0\right| \le 1 + \left|2\sum_{k=1}^{n} \cos kx_0\right| \le \left|\frac{1}{\sin(\frac{x_0}{2})}\right| = M \tag{6}$$

If $f \in BV_0(\mathbb{R})$, then by decomposing Jordan, there exist f_1 and f_2 increasing bounded functions which tend to zero at infinity such that:

$$\sum_{k=-\infty}^{\infty} f(k)e^{ikx_0} = \sum_{k=-\infty}^{\infty} (-f_2(k))e^{-ikx_0} - \sum_{k=-\infty}^{\infty} (-f_1(k))e^{-ikx_0}. \tag{7}$$

Taking in consideration (6) and the Dirichlet criterion for series ([9]), then the series on the right side in (7) are convergent. Therefore, if $f \in BV_0(\mathbb{R})$ then the series

$$\sum_{k=-\infty}^{\infty} f(k)e^{-ikx} \tag{8}$$

is convergent pointwise for each $x \not\equiv 0 \ (mod\ 2\pi)$. On the other hand, taking γ such that $0 < \gamma < \pi$ we have that if $x \in [-\pi,\pi] \setminus (-\gamma,\gamma)$, then

$$\left|\sin\frac{\gamma}{2}\right| \le \left|\sin\frac{x}{2}\right|.$$

This tells us that the convergence of (8) will be uniform on $[-\pi,\pi] \setminus (-\gamma,\gamma)$.

Lemma 2.6. Let $f \in BV_0(\mathbb{R})$. The following relation is satisfied

$$\sup_{0<|x|\le\pi}\left|\sum_{k=-\infty}^{\infty} f(k)e^{ikx} - \int_{-\infty}^{\infty} f(u)e^{iux}\,du\right| \le \frac{1}{2}Var(f;\mathbb{R}).$$

Proof. By Remark 1, the left side of the above expression converges for $x \not\equiv 0 \ (mod\ 2\pi)$. Because of $f \in BV_0(\mathbb{R})$, then by the Riemann-Lebesgue lemma, Lemma 2.4, the integral is defined for $x \neq 0$ and its limit is zero when $|x| \to \infty$. Since

$$\int_{k-\frac{1}{2}}^{k+\frac{1}{2}} e^{iux}\,du = \frac{e^{ikx}}{ix}(2i\sin(\frac{x}{2})),$$

we have that

$$e^{ikx} = \frac{x}{2\sin(\frac{x}{2})}\int_{k-\frac{1}{2}}^{k+\frac{1}{2}} e^{iux}\,du.$$

Therefore

$$\sum_{k=-\infty}^{\infty} f(k)e^{ikx} = \frac{x}{2\sin(\frac{x}{2})}\sum_{k=-\infty}^{\infty}\int_{k-\frac{1}{2}}^{k+\frac{1}{2}} f(k)e^{iux}\,du.$$

Like

$$\int_{-\infty}^{\infty} f(u)e^{iux}\,du = \sum_{k=-\infty}^{\infty}\int_{k-1/2}^{k+1/2} f(u)e^{iux}\,du,$$

and $1 < \frac{x}{2\sin(x/2)}$, for $x \in (-\pi,\pi) \setminus \{0\}$. Then

$$\left|\sum_{k=-\infty}^{\infty} f(k)e^{ikx} - \int_{-\infty}^{\infty} f(u)e^{iux}\,du\right| \le \left|\frac{x}{2\sin(\frac{x}{2})}\sum_{k=-\infty}^{\infty}\int_{k-\frac{1}{2}}^{k+\frac{1}{2}} f(k)e^{iux}\,du - \sum_{k=-\infty}^{\infty}\int_{k-1/2}^{k+1/2} f(u)e^{iux}\,du\right| \tag{9}$$

$$\le \frac{x}{2\sin(x/2)}\left|\sum_{k=-\infty}^{\infty}\int_{k-1/2}^{k+1/2}[f(k)-f(u)]e^{iux}\,du\right|$$

$$+\left[\frac{x}{2\,sin\,(x/2)}-1\right]\left|\int_{-\infty}^{\infty} f\,(u)e^{iux}\;du\right|.$$

For each $n \in \mathbb{N}$:

$$\left|\sum_{k=-n}^{n}\int_{k-1/2}^{k+1/2}(f(k)-f(u))\,du\right|$$

$$\leq \sum_{k=-n}^{n}\int_{k-1/2}^{k+1/2}|f(k)-f(u)|\,du$$

$$=\int_{-1/2}^{1/2}\sum_{k=-n}^{n}|f(k)-f(k+u)|\,du$$

$$\leq Var(f;\mathbb{R}).$$

So that

$$\left|\sum_{k=-\infty}^{\infty}\int_{k-1/2}^{k+1/2}[f(k)-f(u)]e^{iux}\;du\right|\leq Var(f;\mathbb{R}).$$

Then, from the above inequality also (9) and Lemma 2.5, we have that

$$\left|\sum_{k=-\infty}^{\infty} f\,(k)e^{ikx}-\int_{-\infty}^{\infty} f\,(u)e^{iux}\;du\right| \leq \left[\frac{1}{sin\,(x/2)} -\frac{2}{x}\right]Var(f;\mathbb{R}). \qquad (10)$$

The derivative of $g(x)=\frac{1}{sin\,(x/2)}-\frac{2}{x}$ is positive over $(-\pi,\pi)\setminus\{0\}$. Therefore it is increasing and its supremum and infimum are taken over $x=\pi\;,-\pi$, respectively. From this, we must have

$$\sup_{x\in(-\pi,\pi)\setminus\{0\}}\left|\frac{1}{sin\,(x/2)}-\frac{2}{x}\right|\leq\frac{1}{2}.$$

With the last inequality and (10), we obtain the result.□

Remark 2. Let $f_n = f\chi_{[-n,n]}$, where $n \in \mathbb{N}$ and $\chi_{[-n,n]}$ is the characteristic function of $[-n,n]$. If $f \in BV_0(\mathbb{R})$ then $f_n \in BV_0(\mathbb{R})$. Furthermore

$$Var(f_n;\mathbb{R}) \leq Var(f;\mathbb{R}) \quad \text{for all } n \in \mathbb{N}.$$

Therefore, from the previous lemma, for all $n \in \mathbb{N}$:

$$\sup_{0<|x|\leq\pi}\left|\sum_{k=-n}^{n} f\,(k)e^{ikx}-\int_{-n}^{n} f\,(u)e^{iux}\;du\right| \leq \frac{1}{2}Var(f;\mathbb{R}).$$

The above inequality does not change if we multiply the inside terms of the expression by the absolute value of e^{-imx}, where $m \in \mathbb{N}$.

The Dirichlet-Jordan theorem for functions in BV_0 is proved in [8] and [10]. The statement is as follows.

Theorem 2.7. If $f \in BV_0(\mathbb{R})$, then, for each $x \in \mathbb{R}$,

$$\lim_{\substack{M\to\infty\\ \delta\to 0}}\frac{1}{2\pi}\int_{\delta<|\omega|<M} e^{ix\omega}\,\hat{f}(\omega)d\omega = \frac{1}{2}\{f(x+0)+f(x-0)\}. \qquad (11)$$

3. Main theorem

G. Bachman [1] mentions that Josiah Gibbs proved in an article published in *Nature* (vol. 59, p. 606) at 1899, that the Fourier series of the function

$$h(u)=\begin{cases}\frac{-\pi-u}{2}, & -\pi\leq u<0\\ 0, & u=0\\ \frac{\pi-u}{2}, & 0<u<\pi\end{cases} \qquad (12)$$

is

$$\sum_{k=1}^{\infty}\frac{sin\;k\,u}{k}. \qquad (13)$$

This function was used to argue the phenomenon that bears his name. The equality between (13) and (12) is also indicated by Zygmund [11, p. 5]. This fact is used in the proof of the main theorem presented below.

Theorem 3.1. If $f \in BV_0(\mathbb{R})$ and $2f(k)=f(k+0)+f(k-0)$ for all $k \in \mathbb{Z}$, then for $x \not\equiv 0 \;(mod\; 2\pi)$ we have that

$$\sum_{k=-\infty}^{\infty} f\,(k)e^{ikx} = \sum_{k=-\infty}^{\infty}\hat{f}\,(2k\pi-x). \qquad (14)$$

Proof. Let

$$\gamma(x)=\sum_{k=-\infty}^{\infty} f(k)e^{ikx}-\hat{f}(-x) \text{ and } \sigma(x)$$

$$=\sum_{k\neq 0}\hat{f}(2k\pi-x).$$

We will show that the function $s(x)-\sigma(x)$ is continuous on $[-\pi,\pi]$ and its Fourier coefficients are zero. Thus, by the uniqueness theorem , see [12-13], we will have to $s(x)=\sigma(x)$ for $0<|x|\leq\pi$. Since the expressions of both sides in (14) have period 2π, then we will have shown that it is valid for all $x\not\equiv 0 \ (mod\ 2\pi)$.

First analyze the Fourier coefficients $c_m(\gamma)$ of γ. As a result of Remark 2 we have that the succession

$$s_n(x)=\sum_{k=-n}^{n} f(k)e^{ikx}-\widehat{f_n}(-x)$$

is essentially bounded by $\frac{1}{2}Var(f;\mathbb{R})$, and also converges to $\gamma(x)$ almost everywhere when $n\to\infty$. By Lebesgue Dominated Convergence theorem it holds that

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} s_n(x)e^{-imx}\,dx = \int_{-\pi}^{\pi}\gamma(x)e^{-imx}\,dx. \tag{15}$$

As we have seen

$$\int_{-\pi}^{\pi} e^{ix(k-m)}\,dx=\begin{cases}\dfrac{2\sin\pi(k-m)}{(k-m)}=0, & k\neq m\\ 2\pi, & k=m,\end{cases}$$

then, from equality (15),

$$c_m(\gamma)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\gamma(x)e^{-imx}\,dx$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\sum_{k=-\infty}^{\infty} f(k)e^{ikx}-\hat{f}(-x)\right]e^{-imx}\,dx$$

$$=\lim_{n\to\infty}\frac{1}{2\pi}\sum_{k=-n}^{n} f(k)\int_{-\pi}^{\pi} e^{ix(k-m)}\,dx$$

$$-\frac{1}{2\pi}\lim_{\delta\to 0}\int_{0<|\delta|<\pi} e^{-imx}\int_{-\infty}^{\infty} f(u)e^{iux}\,du\,dx \tag{16}$$

$$=f(m)-\frac{1}{2\pi}\lim_{\delta\to 0}\int_{0<|\delta|<\pi} e^{-imx}\hat{f}(-x)dx$$

$$=f(m)-\frac{1}{2\pi}\lim_{\delta\to 0}\int_{0<|\delta|<\pi} e^{imx}\hat{f}(x)dx \tag{17}$$

Now we will analyze the Fourier coefficients of $\sigma(x)$. We denote them as $c_m(\sigma)$. By Lemma 5, taking $2k\pi-x$ instead of ω, and by the equality

$$\frac{1}{(2k\pi-x)}=\frac{1}{2k\pi}+\frac{x}{2k\pi(2k\pi-x)},$$

we have that

$$\sum_{1\leq|k|\leq n}\hat{f}(2k\pi-x)=\sum_{1\leq|k|\leq n}\int_{-\infty}^{\infty} f(u)e^{-iu(2k\pi-x)}du$$

$$=\sum_{1\leq|k|\leq n}\frac{-i}{2k\pi-x}\int_{-\infty}^{\infty} e^{-iu(2k\pi-x)}\,df(u)$$

$$=\sum_{1\leq|k|\leq n}\frac{-i}{2k\pi}\int_{-\infty}^{\infty} e^{-iu(2k\pi-x)}\,df(u) \tag{18}$$

$$+\sum_{1\leq|k|\leq n}\frac{-ix}{2k\pi(2k\pi-x)}\int_{-\infty}^{\infty} e^{-iu(2k\pi-x)}\,df(u).$$

Since on $[-\pi,\pi]$ it holds that

$$\left|\frac{-ix}{2k\pi(2k\pi-x)}\int_{-\infty}^{\infty} e^{-iu(2k\pi-x)}\,df(u)\right|\leq\frac{V(f,\mathbb{R})}{4\pi k^2},$$

then, by the Weierstrass M-test, the second sum on the right side in (18) is uniformly convergent in $[-\pi,\pi]$. In the first sum we have that

$$\sum_{1\leq|k|\leq n}\frac{1}{-i2k\pi}\int_{-\infty}^{\infty} e^{-iu(2k\pi-x)}\,df(u) = -\int_{-\infty}^{\infty} e^{iux}\sum_{k=1}^{n}\frac{\sin 2k\pi u}{k\pi}\,df(u). \tag{19}$$

The series $\sum_{k=1}^{\infty}\frac{\sin 2k\pi u}{k\pi}$ is uniformly convergent on any interval that does not contain points of the form $u=r\pi,\ r\in\mathbb{Z}$, but it is convergent at these points [14]. Furthermore exists $M>0$, [14], such that for all $u\in\mathbb{R}$ and for all $n\in\mathbb{Z}$

$$\left|\sum_{k=1}^{n}\frac{\sin 2k\pi u}{k\pi}\right|\leq M.$$

By the previous comment to this theorem and denoting $\phi(u) = \sum_{k=1}^{\infty} \frac{sin\ 2k\pi u}{k\pi}$,

we obtain that

$$\phi(u) = \begin{cases} \frac{1}{2} - u & 0 < u < 1 \\ 0 & u \in \mathbb{Z} \\ -\frac{1}{2} - u & -1 \le u < 0. \end{cases}$$

Therefore, considering that for any $r \in \mathbb{Z}$: $\sin\ 2\,k\pi(u \pm r) = \sin\ 2\,k\pi u$, we can extend the domain of ϕ to $\mathbb{R}$, as follows:

$$\phi(u) = \begin{cases} \frac{1}{2} - u + [u] & u \notin \mathbb{Z} \\ 0 & u \in \mathbb{Z}. \end{cases}$$

From (19) and by Lebesgue Dominated Convergence theorem:

$$\lim_{n\to\infty} \sum_{1\le|k|<n} \frac{1}{-i2k\pi} \int_{-\infty}^{\infty} e^{-iu(2k\pi - x)}\ df(u)$$

$$= -\int_{-\infty}^{\infty} e^{iux}\ \phi(u) df(u).$$

Thus, we have shown that, on $[-\pi, \pi]$, the partial sum $\sum_{1\le|k|\le n} \hat{f}\,(2k\pi - x)$ is uniformly bounded and converges to a continuous function. Then σ is a continuous function. With respect to its Fourier coefficients, we have

$$c_m(\sigma) = \frac{1}{2\pi} \lim_{n\to\infty} \sum_{1\le|k|\le n} \int_{-\pi}^{\pi} \hat{f}\,(2k\pi - x) e^{-imx}\ dx$$

$$= \frac{1}{2\pi} \lim_{n\to\infty} \int_{\pi\le|x|\le\pi(2n+1)} \hat{f}\,(x) e^{imx}\ dx. \qquad (20)$$

then, from (16) and (20):

$$c_m(s) - c_m(\sigma) = f(m) - \frac{1}{2\pi} \lim_{\delta\to 0} \int_{0<|\delta|<\pi} e^{imx} \hat{f}(x) dx$$

$$- \frac{1}{2\pi} \lim_{n\to\infty} \int_{\pi\le|x|\le\pi(2n+1)} \hat{f}\,(x) e^{imx}\ dx.$$

$$= f(m)$$

$$- \lim_{\substack{n\to\infty \\ \delta\to 0}} \int_{\delta\le|x|\le\pi(2n+1)} \hat{f}\,(x) e^{imx}\ dx.$$

Considering that $2f(k) = f(k+0) + f(k-0)$ for all $k \in \mathbb{Z}$, and by the Theorem 7 we have that $c_m(s) - c_m(\sigma) = 0$, so $s(x) - \sigma(x) = 0$ almost everywhere.

Since the functions s and σ are continuous, then $s(x) = \sigma(x)$ for all $x \not\equiv 0\ (mod\ 2\pi)$.

Acknowledgment

The author expresses his sincere gratitude to Nancy Keranen. This paper was supported partially by VIEP-BUAP, Puebla, Mexico.

References

[1] G. Bachman, L. Narici, E. Beckenstein, Fourier and Wavelet Analysis, Springer-Verlang, New York. Inc., 1991.

[2] M.A. Pinsky, Introduction to Fourier Analysis and Wavelets, CA: Brooks/Cole, Pacific Grove, Australia, 2002.

[3] F.J. Mendoza, J.A. Escamilla, S. Sánchez, Some results about the Henstock-Kurzweil Fourier transform, Mathematica Bohemica 134 (4) (2009) 379-386.

[4] R.M. Trigub, A Generalization of the Euler-Maclaur in Formula Mathematical Notes 61 (2) (1997).

[5] R.G. Bartle, A Modern Theory of Integration, Graduate Studies in Mathematics, Vol. 32, American Mathematical Society, Providence, 2001.

[6] Ch. Swartz, Introduction to Gauge Integrals, World Scientific Publishing Co., Singapore, 2001.

[7] W. Rudin, Principles of Mathematical Analysis, third edition, McGraw-Hill, New York, 1976.

[8] F.J. Mendoza, On Pointwise Inversion of the Fourier Transform of BV_0 Functions, Annals of Functional Analysis 2 (2010) 112-120.

[9] T.M. Apostol, Mathematical Analysis, Addison-Wesley Publishing Company, Reading, Massachusetts, 1991.

[10] R.M. Trigub, E.S. Bellinsky, Fourier Analysis and Approximation of Functions, Kluwer Academic Publishers, Dordrecht, 2004.

[11] A. Zygmund, Trigonometric Series (3rd Edition), Cambridge University Press, Cambridge, 2002.

[12] Y. Katznelson, An Introduction to Harmonic Analysis, Cambridge University Press, Cambridge, 2004.

[13] J. Duondikoitexea, Fourier Analysis, Graduate Studies in Mathematics, American Mathematical Society, Rhode Island, Vol. 29, 1995.

[14] L.D. Kudriavtsev., Curso de Análisis Matemático, Tomo 1, Mir, Moscú, 1983.