Journal of Mathematical Sciences 3 (2016) 120-125
ISSN 2372-5214

Betty Jones & Sisters Publishing
http://www.bettyjonespub.com

Special Paths in Lattice Rectangles

Amal Sharif-Rasslan[1], Manal Gabour[1, 2]

1. Department of Mathematics, The Academic Arab College for Education, Haifa 33145, Israel

2. Department of Mathematics, The Academic Arab College of Beit Berl, Kfar Saba 4490500, Israel

Abstract: We define the concept: a right rectangular path bounded in a $m \times n$ rectangle where $m, n \in \mathbb{N}$. Then we state the problem: what is the relationship between the number of line segments in that path and the rectangle measurements? The process of investigating our problem involves three levels namely numerical, graphical and algebraic. We start at describing the problem, then several examples are treated graphically. Arranging the initial numeric results in a table yields to a primary result regarding our problem. The main result is established through investigating the cases where m, n are and aren't relatively prime. Finaly some generalizations, applications and didactical implications are suggested.

Keywords: Lattice rectangle, right rectangular path, Euclidean Division Algorithm, relatively prime numbers, greater common divisor.

1. Introduction

We investigate the relationship between the measurements of a lattice rectangle and the number of line segments in a special broken line segment which is bounded in the rectangle.

1.1 The Main Problem

Definition 1. A rectangle $ABCD$ is a lattice rectangle, if $AB = m,\ BC = n$ where $m, n \in \mathbb{N}$.

Definition 2. A right rectangular path (RRP) is a "broken line segment" bounded in a lattice rectangle that begins at a vertex, producing a $45°$ angle with an edge that passes through that vertex, ends at a vertex, such that any two successive segments meet vertically at a point on an edge of the rectangle.

Problem 1.

(1) Does a RRP exisit in a lattice rectangle?

In the case where the answer of the above question is yes, then, another question arises:

(2) What is the relation between the number of the line segments in a RRP and the natural rectangle measurements?

Let us begin with some examples, to better understanding the problem. Our following discussion is based on constructing the RRP using the Euclidean Division theorem [1]:

"If $a, b \in Z,\ b > 0$, then there exist unique $q, r \in Z$ such that $a = qb + r, 0 \leq r < b$. Here q is called quotient of the integer division of a by b, and r is called remainder."

In the following examples, we demonstrate an algorithm in order to construct the RRP and later count the number of its segments.

Example 1. Fig 1 describes a 3×5 lattice rectangle.

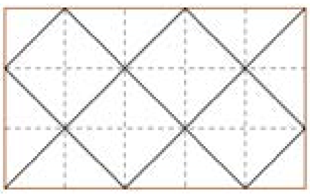

Fig. 1 3×5 rectangle.

Constructing method: via this method we can depend only on the construction of the path, and counting every line segment in it

Explanation: each line segment in the path (Fig. 1), is a hypotenuse of a right and isosceles triangle.

Corresponding author: Amal Sharif-Rasslan, Ph.D., research fields: advanced mathematical thinking, applied mathematics especially in cardiology. E-mail: amalras@macam.ac.il.
Manal Gabour, E-mail: dr.manaljabour@gmail.com.

Since we start at a vertex, the first triangle will have a side of 3 units. Using the "Euclidean Division Algorithm" we get:

$$5 = 1 \cdot 3 + 2 \quad (1.1)$$

therefore the second triangle will have a side of 2 units.

The third triangle has $3 - 2 = 1$ unit as a side.

Applying the above argument two more times we obtain the following:

$$5 - (3 - 2) = 1 \cdot 3 + 1 \quad (1.2)$$

$$5 - (3 - 1) = 1 \cdot 3 + 0 \quad (1.3)$$

Each one of the equations above defines a unique finite sequence (uniqueness is due to the Euclidean Division Algorithm) of natural numbers where each number represents the right side's length of a right and isosceles triangle (Fig. 1). Those sequences are respectively:

Equation (1.1) contributes $3,2$ which are the lengths of the right sides of the two right and isosceles triangles, constructed by the first two segment in the path respectively; equation (1.2) contributes $1,3,1$ which are the lengths of the right sides of the three right and isosceles triangles, constructed by the next three segments in the path; and finally equation (1.3) contributes $2,3$ which are the lengths of the right sides of the two right and isosceles triangles, constructed by the last two segments in the path.

Notice that the sum of the elements in each sequence is 5 and the sum of the last element of any sequence and the first one of its consecutive sequence is 3.

So we get that the number of line segments in the above RRP is $2 + 3 + 2 = 7$, which is the sum of the number of elements in each sequence.

Example 2. Fig. 2 describes a 5×7 lattice rectangle

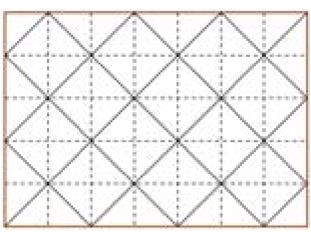

Fig. 2 5×7 rectangle.

In this example we present our division procedure that was proposed previously.

Below we provide the equations obtained by the Euclidean Division Algorithm for the case of 5×7 rectangle, and the sequences related to each equation:

$7 = 1 \cdot 5 + 2 \quad \Longrightarrow \quad 5,2$

$7 - (5 - 2) = 0 \cdot 5 + 4 \quad \Longrightarrow \quad 3,4$

$7 - (5 - 4) = 1 \cdot 5 + 1 \quad \Longrightarrow \quad 1,5,1$

$7 - (5 - 1) = 0 \cdot 5 + 3 \quad \Longrightarrow \quad 4,3$

$7 - (5 - 3) = 1 \cdot 5 + 0 \quad \Longrightarrow \quad 2,5$

Thus, the number of the line segments in the RRP in the 5×7 lattice rectangle equals to the sum of the number of elements in each sequence, i.e. $2 + 2 + 3 + 2 + 2 = 11$; (check it!).

Example 3. 4×13 lattice rectangle

In this example we only present our division procedure that was proposed previously, (check alone by drawing).

$13 = 3 \cdot 4 + 1 \quad \Longrightarrow \quad 4,4,4,1$

$13 - (4 - 1) = 2 \cdot 4 + 2 \quad \Longrightarrow \quad 3,4,4,2$

$13 - (4 - 2) = 2 \cdot 4 + 3 \quad \Longrightarrow \quad 2,4,4,3$

$13 - (4 - 3) = 3 \cdot 4 + 0 \quad \Longrightarrow \quad 1,4,4,4$

Notice that each non-zero residue and each bracket contribute 1 line segment; so, each equation contributes in addition to this number of line segments another number which expressed by the factor of 4. Consequently, the number of line segments in the RRP in the 4×13 lattice rectangle equals to: $(3 + 1) + (1 + 2 + 1) + (1 + 2 + 1) + (1 + 3) = 16$

Remark 1. Notice that in all the above examples we get all the residues: in the first one all residues $mod 3$, in the second example all residues $mod 5$ and in the third one all residues $mod 4$.

Remark 2. Notice that in the first example, the difference between every two successive residues is equivalent to $1 mod 3$ or $2 mod 3$ (depending on the order of the numbers under subtraction); in the second example it is equivalent to $3 mod 5$ or $2 mod 5$, and in the third example it is equivalent to $1 mod 4$ or $3 mod 4$.

Arranging all the above results in a table enables us to conclude the next remark:

rectangle width (m)	rectangle length (n)	number of the path's line segments
3	5	7
5	7	11
4	13	16

Remark 3. The number of the line segments in all the examples above is equal to $m+n-1$.

Notice that the rectangle measurements in all the previous examples are relatively prime. The next example will consider the situation where those measurements haven't this property.

Example 4. 6×10 lattice rectangle

$10=1\cdot 6+4 \quad \Longrightarrow \quad 6,4 \quad \Longrightarrow \quad$ 2 line segments

$10-(6-4)=1\cdot 6+2 \quad \Longrightarrow 2,6,2 \quad \Longrightarrow \; 3$ line segments

$10-(6-2)=1\cdot 6+0 \quad \Longrightarrow \quad 4,6 \quad \Longrightarrow \; 2$ line segments

So, the number of the line segments in this case is 7, the same as the case in the first example (3×5 lattice rectangle).

Comparing the algorithm in example 4 with the algorithm in example 1, we may notice the following:

1. There are the same number of equations;

2. The equations of the division algorithm in example 4 may be obtained after multiplying by 2, each equation in the algorithm of example 1.

Therefore, it can be obtained that both paths posses the same number of line segments, but yet that number in the 6×10 rectangle, differs from what we get by the formula $m+n-1$.

Thus, studying our main problem will be done through distinguishing between two subcases:

I. $(m,n)=1, (m$ and n are relatively prime).

II. $(m,n)\neq 1, (m$ and n are not relatively prime).

2. The Case $(m,n)=1$

Although restricting to the case $(m,n)=1$, several coming results will be valid also otherwise, in other words the restriction $(m,n)=1$ will be mentioned only whenever it is fundamental. Without less of generality let us suppose that $m>n$.

Proposition 1. Every lattice rectangle has a unique RRP up to plane isometries.

Proof. Let $ABCD$ be a $m\times n$ lattice rectangle, finding the RRP bounded in it is due to the division algorithm that was demonstrated by the examples above. That algorithm can be generalized to the form:

$m=q_1n+r_1$

$m-(n-r_1)=q_2n+r_2$

$\vdots$

$m-(n-r_{k-1})=q_kn+r_k$

according to the Euclidean Division Algorithm we know that at each $i-$step, q_i and r_i exist uniquely. Thus the RRP also exists uniquely.

Remark 4. Notice that getting a RRP is equivalent to getting $r_k=0$ for some $k\in\mathbb{N}$.

Proposition 2. Given a $m\times n$ lattice rectangle, then the difference between any two successive residues in the division algorithm is constant mod n.

Proof. In this algorithm, the i^{th} step is

$m-(n-r_{i-1})=q_in+r_i$

and the $(i+1)$ step is

$m-(n-r_i)=q_{i+1}n+r_{i+1}$

by subtraction we get:

$$-(n-r_i)+(n-r_{i-1})=(q_{i+1}-q_i)n+(r_{i+1}-r_i)$$
$$\Rightarrow r_i-r_{i-1}=(q_{i+1}-q_i)n+(r_{i+1}-r_i)$$
$$\equiv (r_{i+1}-r_i)(modn).$$

Proposition 3. Given a $m\times n$ lattice rectangle, then the difference between any two successive residues in the division algorithm is equivalent to $r_1 modn$.

Proof. Considering the first two steps of the division algorithm:

$m=q_1n+r_1$

$m-(n-r_1)=q_2n+r_2$

we get by subtraction that

$-n+r_1=(q_2-q_1)n+r_2-r_1 \Leftrightarrow r_2-r_1\equiv r_1(modn)$.

Combining this result to that of the pevious proposition yeilds the desired result.

Proposition 4. Given a $m \times n$ lattice rectangle, the division algorithm ends within n consecutive steps.

Proof. According to the Euclidean Division Algorithm there exist $q_1, r_1 \in Z$ such that $m = q_1 n + r_1$.

If we define $\alpha = \Delta r_i = r_{i+1} - r_i$, then it will be sufficient to prove that:

$r_1 + (n-1)\alpha \equiv 0(modn)$.

This congruency is easily verified by substituting $\alpha = r_1$.

Conclusion 1. Each $m \times n$ lattice rectangle where $(m,n) = 1$, has a RRP.

Now we prove that for a $m \times n$ lattice rectangle, n is the minimal number of steps needed for ending the division algorithm.

Proposition 5. There is no $k \in Z$, $0 < k < n-1$ such that $r_1 + k\alpha \equiv 0(modn)$.

Proof. Considering the first step of the algorithm : $m = q_1 n + r_1$, we get that whenever $(m,n) = 1$, it is necessary that also $(r_1, n) = 1$.

Now let us suppose that there exists $k \in Z$, $0 < k < n-1$ such that $r_1 + k\alpha \equiv 0(modn)$. After the substitution $\alpha = r_1$, we obtain: $r_1 + kr_1 \equiv 0(modn) \Leftrightarrow r_1(1+k) \equiv 0(modn)$, since $(r_1, n) = 1$, there exists $t \in \mathbb{N}$ such that $1 + k = tn$, which contradicts the assumption that $k < n-1$.

Theorem 1. If ABCD is a $m \times n$ lattice rectangle where $m > n$ and $(m,n) = 1$, then the number of the line segments in the RRP bounded in it is $m + n - 1$.

Proof. There are n division steps needed to obtain the number of the line segments in the RRP,

$m = q_1 n + r_1$

$m - (n - r_1) = q_2 n + r_2$

$\vdots$

$m - (n - r_{n-2}) = q_{n-1} n + r_{n-1}$

$m - (n - r_{n-1}) = q_n n$.

The number of the line segment in the path may be obtained by the summation:

$$\sum_{k=1}^{n-1} 1 + \sum_{i=1}^{n} q_i + \sum_{k=1}^{n-1} 1 = 2(n-1) + \sum_{i=1}^{n} q_i \qquad (2.1)$$

In addition, adding the n equations in the above algorithm leads to:

$m + (m-n)(n-1) = n\sum_{i=1}^{n} q_i \Leftrightarrow n(m-n+1) = n\sum_{i=1}^{n} q_i \Leftrightarrow (m-n+1) = \sum_{i=1}^{n} q_i$.

After substituting this sum in equation (2.1) we get:

$\sum_{k=1}^{n-1} 1 + \sum_{i=1}^{n} q_i + \sum_{k=1}^{n-1} 1 = 2(n-1) + \sum_{i=1}^{n} q_i = 2(n-1) + m - n + 1 = m + n - 1$.

3. The Case (m,n)≠ 1

Without less of generality let us suppose that $m \geq n$.

Let $d = (m,n)$ (d is the greatest common divisor of the numbers m, n). Applying the Euclidean Division Algorithm on the numbers $\frac{m}{d}, \frac{n}{d}$ we get a unique quotient q, and a unique remainder r (where $q \in \mathbb{Z}$, $0 \leq r < \frac{n}{d} - 1$) such that:

$$\frac{m}{d} = q \cdot \frac{n}{d} + r \qquad (3.1)$$

multiplying the last equation by d,

$$m = qn + dr \qquad (3.2)$$

hence the following remark is verified.

Remark 5. When dividing m by n we get the same quotient but d- times the remainder obtained when $\frac{m}{d}$ is divided by $\frac{n}{d}$.

Now we are ready to state the main theorem of this section.

Theorem 1. Let $ABCD$ be a $m \times n$ lattice rectangle where $(m,n) \neq 1$. The number of the line segments in the RRP bounded in it equals to

$\frac{m}{(m,n)} + \frac{n}{(m,n)} - 1$

Proof. Let $(m,n) = d$. The division algorithm related to the $m \times n$ rectangle can be obtained by that related to the $\frac{m}{d} \times \frac{n}{d}$ rectangle after multiplying each equation by d. Following remark 5, through knowing that each nonzero remainder contributes 1 line segment to the path, leads to the conclusion that the $m \times n$ rectangle and the $\frac{m}{d} \times \frac{n}{d}$ rectangle both have the same number of line segments in each RRP bounded in each of them. Since $(\frac{m}{d}, \frac{n}{d}) = 1$, by Theorem 1 in section 2 we get that the number of those line segments is $\frac{m}{(m,n)} + \frac{n}{(m,n)} - 1$.

Remark 6. Notice that when $(m, n) = 1$ then $\frac{m}{(m,n)} + \frac{n}{(m,n)} - 1 = m + n - 1$. Therefore Theorem 1 in section 2 and Theorem 1 in section 3 can be combined to one main theorem that includes the first one as a private case.

Theorem 2. Let $ABCD$ be a $m \times n$ lattice rectangle. The number of the line segments in the RRP bounded in it equals to $\frac{m}{(m,n)} + \frac{n}{(m,n)} - 1$.

4. Generalizations

4.1 Rectangle with rational measurements:

Let $ABCD$ be a rectangle, where $AB = p$, $BC = q$ and $p, q \in \mathbb{Q}$.

Without less of generality, we may consider $p = \frac{m_p}{n_p}$ and $q = \frac{m_q}{n_q}$ as reduced fractions (m_p, m_q, n_p, $n_q \in \mathbb{N}$). In this case, let $[n_p, n_q]$ be the smallest common multiple of the denominators of p and q. It is obvious that the given rectangle can be considered as a lattice rectangle with $\frac{1}{[n_p,n_q]}$ as its unit. Thus, the number of the line segments in the RRP bounded in it is: $\frac{p[n_p,n_q] + q[n_p,n_q]}{(p[n_p,n_q],q[n_p,n_q])} - 1$.

4.2 The generalized path in a parallelogram:

The main problem above, may be generalized as follows: Does a parallelogram contain an analogical path to the RRP in a lattice rectangle? In other words, does a parallelogram contain a special path which begins and ends at two different vertices and in some way is analogical to the RRP? If so, does the number of segments in that path is still related to the parallelogram measurements?

The answer to the last questions is yes. Some parallelograms contain a path analogical to that of the lattice rectangle; it can be produced by a conditional process.

Proposition 6. Let $ABCD$ be a parallelogram where α is its acute angle, $AB = \frac{n}{sin\ \alpha}$, $BC = m$, where $m, n \in \mathbb{N}$. Then, there exists a unique path connecting two different vertices through crossing the sides of the parallelogram, of the length $\frac{m}{(m,n)} + \frac{n}{(m,n)} - 1$.

The verification of the last proposition bases on the property that every parallelogram can be transformed to a rectangle and vise versa, by the shear affine transformation in which all points along a given line l remain fixed while other points are shifted parallel to l by a distance proportional to their perpendicular distance from l (in our case the line l is one edge of the parallelogram or the rectangle), Fig. 3. Shearing a plane figure does not change its area; in addition, the number of sides and vertices of a polygon are invariants under shear affine transformation. Notice that by shear affine transformation the given parallelogram is transformed to a lattice rectangle.

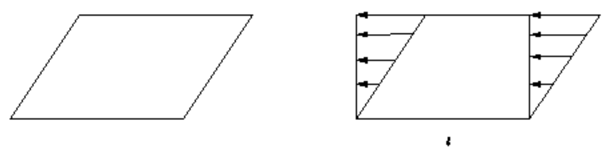

Fig. 3 5×7 rectangle.

4.3 Applications

Looking for applications for our main result reveals to some possiblilies such as building conditional engineering nets.

Let A and B be any two different points in the space.

If it is impossible to reach the point B from A throught the segment AB, then, we may consider the points A and B as two vertices of a rectangle. Thus, using our main result above, enables us to find the shortest path. beginning at A and ending at B, throught crossing the sides of the rectangle.

Remark 7. If the measeurments of the obtained rectangle are not natural, then we may approximate its measeurements by two rational numbers, using the density property of rationals [2].

5. Didactical Implications

The above problem can be given to students as an open-ended planar problem, in which they are supposed to explore the relation between the measurments of a lattice rectangle and the number of the line segments in the RRP bounded in it. It is worthy to mention that the above problem is based on the work of Brousseau [3] and Vergnaud [4], in which the focus is on the interaction student-teacher-know is rewarding and meaningful for both the student and the teacher. Moreover, this problem enables students to learn according to the formal structure of mathematics which is characterized by understanding the problem, defining its elements, investigating private cases, generalization, making conjectures and proving them [5].

References

[1] K.H. Rosen, Elementary Number Theory, Sixth Edition, Pearson New International Edition, 2013.

[2] E. Zakon, Mathematical Analysis, Vol. I. Published by The Trillia Group, West Lafayette, Indiana, USA, 2004.

[3] G. Brousseau, A Review of Theory of Didactical Situations in Mathematics, Didactique des Mathematiques, (1970-1990) translated Nicolas Balacheff. Journal of Research in Mathematics Education, 1998, Vol 29, No. 5.

[4] G. Vergnaud, Morphismes Fondamentaux dans les Processus de Conceptualisation. Les Sciences Cognitive en Debat EditionS du CNRS, 1991.

[5] National Council of Teachers of Mathematics, Professional Standards for Teaching Mathematics. USA: NCTM, 1996.