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Hurwitz-Lerch Zeta Function with Order 1

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Abstract: In this paper, we firstly introduce the Hurwitz-Lerch zeta function with order 1 which is a special case of the Hurwitz-Lerch zeta function. Then, by using an integral representation related to Hurwitz-Lerch zeta function, we obtain some new identities. Also we give some applications to polynomials associated with Harmonic numbers.

Key words: Hurwitz-Lerch zeta function, Hurwitz zeta function, Riemann zeta function, Harmonic numbers, Leibniz derivative formula.

1. Introduction, Definitions and Preliminaries

Throughout this article, we use the following standard notations:

$\mathbb{N}$ denotes the set of natural numbers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers. Also,

$$\mathbb{N}_0 = \{0,1,2,3,\cdots\} = \mathbb{N} \cup \{0\}$$

and

$$\mathbb{Z}^- = \{-1,-2,-3,\cdots\} = \mathbb{Z}_0^- \setminus \{0\}.$$

The Hurwitz-Lerch zeta function $\Phi(z,s,w)$ is defined by (cf. [1], [2], [3])

$$\Phi(z,s,w) = \sum_{n=0}^{\infty} \frac{z^n}{(n+w)^s} \tag{1.1}$$

$$(w \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ s \in \mathbb{C} \text{ when } |z| < 1;\ Re(s) > 1 \text{ when } |z| = 1).$$

The Hurwitz-Lerch zeta function $\Phi(z,s,w)$ can be continued meromorphically to the whole complex s-plane, except for a simple pole at $s = 1$ with its residue 1.

Riemann zeta function and Hurwitz zeta function are the special cases of the Hurwitz-Lerch zeta function: $\zeta(s) = \Phi(1,s,1)$ and $\zeta(s,w) = \Phi(1,s,w)$ where (cf. [4], [5])

$$\zeta(s) \sum_{n=0}^{\infty} \frac{1}{n^s} \ ,\ (Re(s) > 1)$$

and

$$\zeta(s,w) = \sum_{n=0}^{\infty} \frac{1}{(n+w)^s} \ ,\ (Re(s) > 1,\ w \in \mathbb{C} \setminus \mathbb{Z}_0^-).$$

Also many authors have studied on generalizations of Riemann zeta function (cf. [6], [7], [8], [9]).

The Hurwitz-Lerch zeta function contains, as its special cases, the other important functions of Analytic Function Theory as the polylogarithmic function (or de Jonquiére's function) $Li_s(z)$ (cf. [10]):

$$Li_s(z) = \sum_{n=0}^{\infty} \frac{z^n}{n^s} = z\Phi(z,s,1)$$

$$(s \in \mathbb{C} \text{ when } |z| < 1;\ Re(s) > 1 \text{ when } |z| = 1)$$

and the Lerch zeta function $l_s(\xi)$ (cf. [11]):

$$l_s(\xi) = \sum_{n=0}^{\infty} \frac{e^{2\pi i n \xi}}{n^s} = e^{2\pi i \xi}\Phi(e^{2\pi i \xi}, s, 1)$$

$$(\xi \in \mathbb{R};\ Re(s) > 1).$$

By writing the Eulerian integral in the form of

$$\Gamma(z) = s^z \int_0^{\infty} t^{z-1} e^{-st} dt \ ,\ (Re(z) > 0;\ Re(s) > 0)$$

and using the expansion of series

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$$\frac{1}{1-ze^{-t}}=\sum_{n=0}^{\infty} z^m e^{-mt} \quad , \ (|z|<e^t),$$

we can deduce the following integral representation (cf. [12]) from (1.1):

$$\Phi(z,s,w)=\frac{1}{\Gamma(s)}\int_0^{\infty}\frac{t^{s-1}e^{-wt}}{1-ze^{-t}}dt \qquad (1.2)$$

$$=\frac{1}{\Gamma(s)}\int_0^{\infty}\frac{t^{s-1}e^{-(w-1)t}}{e^t-z}dt$$

$(Re(w)>0;\ Re(s)>0$ when $|z|\le 1,\ z\ne 1;$
$Re(s)>1$ when $|z|=1)$

by noting that

$$\frac{z^n}{(n+w)^s}=\frac{1}{\Gamma(s)}\int_0^{\infty} e^{-wt}t^{s-1}(ze^{-t})^n dt \quad ,$$

$(Re(w)>0,\ Re(s)>0).$

When t in (1.2) is replaced by $-\log\ t$, (1.2) is also written in an equivalent form:

$$\Phi(z,s,w)=\frac{(-1)^{s-1}}{\Gamma(s)}\int_0^1\frac{(log\ \ t)^{s-1}t^{w-1}}{1-zt}dt.$$

For $s=1$, we have

$$\Phi(z,1,w)=\int_0^1\frac{t^{w-1}}{1-zt}dt \ , \qquad (1.3)$$

$(Re(w)>0;\ |z|\le 1,\ z\ne 1).$

Definition 1. s is called as "an order of the Hurwitz-Lerch zeta function $\Phi(z,s,w)$" defined by (1.1).

In [13], Aygunes obtained some identities related to Hurwitz-Lerch zeta function with order 1. In Section 2, we focus on the Hurwitz-Lerch zeta function with order 1.

Just as the Riemann zeta function and the Hurwitz zeta function, the Hurwitz-Lerch zeta function is also useful function. The Hurwitz-Lerch zeta function $\Phi(z,s,w)$ is very important not only in the theory of Analytic Number Theory but also in many branch of Mathematics and Mathematical Physics. Recently, many authors has studied the function $\Phi(z,s,w)$ in (1.1) in various ways:

In [12], Choi investigated the analytic continuation of the Lerch zeta function and also functional equation for the Lerch zeta function.

Also, Bin-Saad defined the generalized double zeta function (cf. [14]):

$$\zeta_\lambda^\mu(x,y;s,a)=\frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-1}\frac{e^{-(a-1)t}(1-xe^{-\lambda t})^{-\mu}}{(e^{-t}-y)}dt$$

where $Re(\mu)>0$, $Re(\lambda)>0$ and either $|x|\le 1$, $|y|\le 1,\ x\ne 1,\ y\ne 1$. ζ_λ^μ is derived directly from the corresponding integral representation of the Hurwitz-Lerch zeta function $\Phi(z,s,w)$.

We note that

$$\zeta_1^\mu(xy,y;s,w)=(1-x)^{-\mu}\Phi(y,s,w)$$

where $\lambda=1,\ |x|<1$ and $|y|<1$.

2. Main Results

In this section, for our main results, we use a special case of the Hurwitz-Lerch zeta function. Then we obtain some functional equations by using the special integral representation in (1.3).

Let $M,N\in\mathbb{N}_0\setminus\{0\}$. Then,

$$\Phi(Nz,1,M)=\int_0^1\frac{t^{M-1}}{1-Nzt}dt \qquad (2.1)$$

When t in (2.1) is replaced by t/N, we have

$$\Phi(Nz,1,M)=\frac{1}{N^M}\int_0^N\frac{t^{M-1}}{1-zt}dt \quad .$$

Denote such a partition P, that is $P=\{0,1,2,\cdots,N\}$. For $k\in\mathbb{N}_0$, we integrate on $[k,k+1]$:

$$\int_0^N\frac{t^{M-1}}{1-zt}dt=\sum_{k=0}^{N-1}\int_k^{k+1}\frac{t^{M-1}}{1-zt}dt$$

and replacing t by $t+k$, we have

$$\int_0^N\frac{t^{M-1}}{1-zt}dt=\sum_{k=0}^{N-1}\int_0^1\frac{(t+k)^{M-1}}{1-z(t+k)}dt.$$

Then, we get

$$N^M \Phi(Nz,1,M) = \sum_{k=0}^{N-1} \int_0^1 \frac{(t+k)^{M-1}}{1-z(t+k)} dt$$

$$= \sum_{k=0}^{N-1} \int_0^1 \frac{1}{1-z(t+k)} \sum_{j=0}^{M-1} \binom{M-1}{j} t^j k^{M-1-j} dt$$

$$= \sum_{k=0}^{N-1} \frac{1}{1-zk} \int_0^1 \frac{1}{1-\frac{zt}{1-zk}} \sum_{j=0}^{M-1} \binom{M-1}{j} t^j k^{M-1-j} dt$$

$$= \sum_{k=0}^{N-1} \sum_{j=0}^{M-1} \binom{M-1}{j} \frac{k^{M-1-j}}{1-zk} \int_0^1 \frac{t^j}{1-\frac{zt}{1-zk}} dt.$$

Consequently, we arrive at the following theorem:

Theorem 1. Let $M, N \in \mathbb{N}_0 \setminus \{0\}$ and $|z| < 1/N$. Then,

$$\Phi(Nz,1,M) = \frac{1}{N^M} \sum_{k=0}^{N-1} \sum_{j=0}^{M-1} \binom{M-1}{j} \frac{k^{M-1-j}}{1-zk} \Phi\left(\frac{z}{1-zk},1,j+1\right) \tag{2.2}$$

Let $n \in \mathbb{N}_0$. Also, there is an integral representation for the n-th partial derivative of Hurwitz-Lerch zeta function $\Phi(z,1,w)$ with respect to z:

$$\frac{\partial^n}{\partial z^n} \Phi(z,1,w) = \frac{\partial^n}{\partial z^n} \int_0^1 \frac{t^{w-1}}{1-zt} dt$$

$$= \int_0^1 t^{w-1} \left\{ \frac{\partial^n}{\partial z^n} \left(\frac{1}{1-zt} \right) \right\} dt$$

$$= \int_0^1 t^{w-1} \frac{n!\, t^n}{(1-zt)^{n+1}} dt$$

Then, we have

$$\frac{1}{n!} \frac{\partial^n}{\partial z^n} \Phi(z,1,w) = \int_0^1 \frac{t^{w+n-1}}{(1-zt)^{n+1}} dt$$

where

$$\frac{\partial^n}{\partial z^n} \Phi(z,1,w) = \frac{\partial^n}{\partial z^n} \sum_{m=0}^{\infty} \frac{z^m}{m+w}$$

$$= \sum_{m=n}^{\infty} \frac{m!}{(m-n)!} \frac{z^{m-n}}{m+w}.$$

Let $R \in \mathbb{N}_0 \setminus \{0\}$. We take the $(R-1)$-th derivative of equation (2.2):

$$\frac{d^{R-1}}{dz^{R-1}} \Phi(Nz,1,M)$$

$$= \frac{1}{N^M} \sum_{k=0}^{N-1} \sum_{j=0}^{M-1} \binom{M-1}{j} k^{M-1-j} \sum_{i=0}^{R-1} \binom{R-1}{j}$$

$$\times \frac{(R-i-1)!\, k^{R-i-1}}{(1-zk)^{R-i}} \times \frac{d^i}{dz^i} \Phi\left(\frac{z}{1-zk},1,j+1\right)$$

or

$$\frac{d^{R-1}}{dz^{R-1}} \Phi(Nz,1,M)$$

$$= \frac{1}{N^M} \sum_{i=0}^{R-1} \sum_{k=0}^{N-1} \sum_{j=0}^{M-1} \binom{M-1}{j} \frac{(R-1)!\, k^{M+R-i-j-2}}{i!\,(1-zk)^{R-i}}$$

$$\times \frac{d^i}{dz^i} \Phi\left(\frac{z}{1-zk},1,j+1\right)$$

where Leibniz derivative formula is given by (cf. [15])

$$D^n(f.g) = \sum_{k=0}^{n} \binom{n}{k} (D^{n-k} f)(D^k g).$$

Remark 1. As a special case, by putting $R = 2$, we have

$$N^{M+1} \Phi'(Nz,1,M) = \sum_{k=0}^{N-1} \sum_{j=0}^{M-1} \binom{M-1}{j} k^{M-1-j}$$

$$\times \left\{ \frac{k}{(1-zk)^2} \Phi\left(\frac{z}{1-zk},1,j+1\right) + \frac{1}{(1-zk)^3} \Phi'\left(\frac{z}{1-zk},1,j+1\right) \right\}.$$

For preparing our second theorem, we suppose that $a > 0$. We recall that

$$\Phi(z,1,M)=\int_0^1 \frac{t^{M-1}}{1-zt}dt$$

Replacing t by $t-a$ in the above integral, we get

$$\Phi(z,1,M)=\int_a^{a+1} \frac{(t-a)^{M-1}}{1-z(t-a)}dt$$

$$=\int_a^{a+1} \frac{(t-a)^{M-1}}{(1+az)-zt}dt$$

$$=\frac{1}{1+az}\int_a^{a+1} \frac{(t-a)^{M-1}}{1-\left(\frac{z}{1+az}\right)t}dt$$

We set

$$\int_a^{a+1} \frac{(t-a)^{M-1}}{1-\left(\frac{z}{1+az}\right)t}dt$$

$$=\int_0^{a+1} \frac{(t-a)^{M-1}}{1-\left(\frac{z}{1+az}\right)t}dt \qquad (2.3)$$

$$-\int_0^{a} \frac{(t-a)^{M-1}}{1-\left(\frac{z}{1+az}\right)t}dt$$

Then, we define

$$I(a,M,z)=I_1(a,M,z)+I_2(a,M,z),$$

$$I_1(a,M,z)=\int_0^{a+1} \frac{(t-a)^{M-1}}{1-\left(\frac{z}{1+az}\right)t}dt$$

and

$$I_2(a,M,z)=\int_0^{a} \frac{(t-a)^{M-1}}{1-\left(\frac{z}{1+az}\right)t}dt.$$

Replacing t by $t(a+1)$ for $I_1(a,M,z)$, we get

$$I_1(a,M,z)=\int_0^{a+1} \frac{(t-a)^{M-1}}{1-\left(\frac{z}{1+az}\right)t}dt$$

$$=\int_0^{1} \frac{(t(a+1)-a)^{M-1}}{1-\left(\frac{az+z}{1+az}\right)t}dt. \qquad (2.4)$$

Replacing t by ta for $I_2(a,M,z)$, we get

$$I_2(a,M,z)=\int_0^{a} \frac{(t-a)^{M-1}}{1-\left(\frac{z}{1+az}\right)t}dt$$

$$=\int_0^{1} \frac{(ta-a)^{M-1}}{1-\left(\frac{az}{1+az}\right)t}dt. \qquad (2.5)$$

From (2.4) and (2.5), we arrange (2.3) as follows:

$$I(a,M,z)=\int_a^{a+1} \frac{(t-a)^{M-1}}{1-\left(\frac{z}{1+az}\right)t}dt$$

$$=I_1(a,M,z)+I_2(a,M,z)$$

$$=(a+1)^M\int_0^{1} \frac{\left(t-\frac{a}{a+1}\right)^{M-1}}{1-\left(\frac{az+z}{az+1}\right)t}dt$$

$$-a^M\int_0^{1} \frac{(t-1)^{M-1}}{1-\left(\frac{az}{az+1}\right)t}dt.$$

We define

$$I_3(a,M,z)=\int_0^{1} \frac{\left(t-\frac{a}{a+1}\right)^{M-1}}{1-\left(\frac{az+z}{az+1}\right)t}dt$$

t

and

$$I_4(a,M,z)=\int_0^{1} \frac{(t-1)^{M-1}}{1-\left(\frac{az}{az+1}\right)t}dt.$$

By using the binomial expansion for $I_3(a,M,z)$, we have

$I_3(a,M,z)$

$$= \int_0^1 \frac{1}{1-\left(\frac{az+z}{az+1}\right)t} \sum_{k=0}^{M-1} \binom{M-1}{k} t^k \left(-\frac{a}{a+1}\right)^{M-k-1} dt$$

$$= \sum_{k=0}^{M-1} \binom{M-1}{k} \left(-\frac{a}{a+1}\right)^{M-k-1} \int_0^1 \frac{t^k dt}{1-\left(\frac{az+z}{az+1}\right)t}$$

$$= \sum_{k=0}^{M-1} \binom{M-1}{k} \left(-\frac{a}{a+1}\right)^{M-k-1} \Phi\left(\frac{az+z}{az+1}, 1, k+1\right).$$

By using the binomial expansion for $I_4(a,M,z)$, we have

$I_4(a,M,z)$

$$= \int_0^1 \frac{1}{1-\left(\frac{az}{az+1}\right)t} \sum_{j=0}^{M-1} \binom{M-1}{j} (-1)^{M-1-j} t^j dt$$

$$= \sum_{j=0}^{M-1} \binom{M-1}{j} (-1)^{M-1-j} \int_0^1 \frac{t^j dt}{1-\left(\frac{az}{az+1}\right)t}$$

$$= \sum_{j=0}^{M-1} \binom{M-1}{j} (-1)^{M-1-j} \Phi\left(\frac{az}{az+1}, 1, j+1\right).$$

Then, we arrive

$$\Phi(z,1,M) = \frac{1}{1+az} \sum_{k=0}^{M-1} \binom{M-1}{k} (-1)^{M-k-1}$$

$$\left\{ \begin{matrix} (a+1)^M \left(-\frac{a}{a+1}\right)^{M-k-1} \Phi\left(\frac{az+z}{az+1}, 1, k+1\right) \\ -a^M (-1)^{M-k-1} \Phi\left(\frac{az}{az+1}, 1, k+1\right) \end{matrix} \right\}.$$

By using the above equation, we obtain the following theorem:

Theorem 2. Let $a > 0$, $M \in \mathbb{N}_0 \setminus \{0\}$ and $|z| < 1$. Then, we have

$$\Phi(z,1,M) = \frac{a^M}{1+az} \sum_{k=0}^{M-1} \binom{M-1}{k} (-1)^{M-k-1}$$

$$\left\{ \begin{matrix} \left(\frac{a+1}{a}\right)^{k+1} \Phi\left(\frac{az+z}{az+1}, 1, k+1\right) \\ -\Phi\left(\frac{az}{az+1}, 1, k+1\right) \end{matrix} \right\}.$$

For the other theorem, we suppose that $b > a > 0$, $|z| < 1/a$ and $|z| < 1/b$. We set

$$\int_a^b \frac{t^{M-1}}{1-zt} dt = \int_0^b \frac{t^{M-1}}{1-zt} dt - \int_0^a \frac{t^{M-1}}{1-zt} dt.$$

We define

$$I_b(z,M) = \int_0^b \frac{t^{M-1}}{1-zt} dt$$

and

$$I_a(z,M) = \int_0^a \frac{t^{M-1}}{1-zt} dt$$

Replacing t by tb for $I_b(z,M)$, we get

$$I_b(z,M) = b^M \Phi(bz,1,M).$$

Replacing t by ta for $I_a(z,M)$, we get

$$I_a(z,M) = a^M \Phi(az,1,M).$$

Then, we have

$$\int_a^b \frac{t^{M-1}}{1-zt} dt = b^M \Phi(bz,1,M) - a^M \Phi(az,1,M). \quad (2.6)$$

On the other hand, we define $I_{a,b}(z,M)$ by

$$I_{a,b}(z,M) = \int_a^b \frac{t^{M-1}}{1-zt} dt$$

and replacing t by $t+a$ for $I_{a,b}(z,M)$, we get

$$I_{a,b}(z,M) = \int_0^{b-a} \frac{(t+a)^{M-1}}{1-za-zt} dt$$

$$= \frac{1}{1-za} \int_0^{b-a} \frac{(t+a)^{M-1}}{1-\left(\frac{z}{1-za}\right)t} dt.$$

Replacing t by $t(b-a)$ in the above integral, we have

$$I_{a,b}(z,M)$$

$$=\frac{1}{1-za}\int_0^1 \frac{(t(b-a)+a)^{M-1}(b-a)}{1-\left(\frac{z(b-a)}{1-za}\right)t}dt$$

$$=\frac{(b-a)^M}{1-za}\int_0^1 \frac{\left(t+\frac{a}{b-a}\right)^{M-1}}{1-\left(\frac{z(b-a)}{1-za}\right)t}dt$$

$$=\frac{(b-a)^M}{1-za}\int_0^1 \frac{1}{1-\left(\frac{z(b-a)}{1-za}\right)t}\sum_{j=0}^{M-1}\binom{M-1}{j}$$

$$\left(\frac{a}{b-a}\right)^{M-j-1} t^j dt$$

$$=\frac{(b-a)^M}{1-za}\sum_{j=0}^{M-1}\binom{M-1}{j}\left(\frac{a}{b-a}\right)^{M-j-1}$$

$$\int_0^1 \frac{t^j}{1-\left(\frac{z(b-a)}{1-za}\right)t}dt.$$

Then, we have

$$I_{a,b}(z,M)=\frac{(b-a)^M}{1-za}\sum_{j=0}^{M-1}\binom{M-1}{j}$$

$$\left(\frac{a}{b-a}\right)^{M-j-1}\Phi\left(\frac{z(b-a)}{1-za},1,j+1\right). \tag{2.7}$$

From (2.6) and (2.7), we arrive at the following theorem:

Theorem 3. Let $b>a>0$, $|z|<1/a$ and $|z|<1/b$. Then, we have

$$b^M\Phi(bz,1,M)-a^M\Phi(az,1,M)$$

$$=\frac{(b-a)^M}{1-za}\sum_{j=0}^{M-1}\binom{M-1}{j}\left(\frac{a}{b-a}\right)^{M-j-1}$$

$$\Phi\left(\frac{z(b-a)}{1-za},1,j+1\right).$$

Let $a,b\in\mathbb{N}_0\setminus\{0\}$. From Theorem 1, we have

$$a^M\Phi(az,1,M)$$

$$=\sum_{k=0}^{a-1}\sum_{j=0}^{M-1}\binom{M-1}{j}\frac{k^{M-j-1}}{1-zk}\Phi\left(\frac{z}{1-zk},1,j+1\right) \tag{2.8}$$

and

$$b^M\Phi(bz,1,M)$$

$$=\sum_{k=0}^{b-1}\sum_{j=0}^{M-1}\binom{M-1}{j}\frac{k^{M-j-1}}{1-zk}\Phi\left(\frac{z}{1-zk},1,j+1\right). \tag{2.9}$$

From equation (2.8), equation (2.9) and Theorem 3, we obtain the following theorem:

Theorem 4. Let $a,b,M\in\mathbb{N}_0\setminus\{0\}$ and $b>a>0$. Then, we have

$$\sum_{k=\mathrm{a}}^{b-1}\sum_{j=0}^{M-1}\binom{M-1}{j}\frac{k^{M-j-1}}{1-zk}\Phi\left(\frac{z}{1-zk},1,j+1\right)$$

$$=\frac{(b-a)^M}{1-za}\sum_{j=0}^{M-1}\binom{M-1}{j}\left(\frac{a}{b-a}\right)^{M-j-1}\Phi\left(\frac{z(b-a)}{1-za},1,j+1\right)$$

where $|z|<1/a$ and $|z|<1/b$.

3. Applications to Polynomials associated with Harmonic Numbers

In this section, we fistly define the polynomials associated with Harmonic numbers as follows:

$$A_{M-1}(z)=\sum_{k=1}^{M-1}\frac{z^k}{k}$$

where $M\in\mathbb{N}_0\setminus\{0,1\}$.

Let $|z|<1$. Then, we explicitly expand the Hurwitz-Lerch zeta function with order 1:

$$\Phi(z,1,M)=\sum_{n=0}^{\infty}\frac{z^n}{n+M}$$

$$= \sum_{n=M}^{\infty} \frac{z^{n-M}}{n} = z^{-M}\left\{\left(\sum_{n=1}^{\infty} \frac{z^n}{n}\right) - \left(\sum_{k=1}^{M-1} \frac{z^k}{k}\right)\right\}$$
$$= -z^{-M}\{\log(1-z) + A_{M-1}(z)\}.$$

We note that $A_0(z) = 0$ for $M = 1$. By using this expansion, we obtain some corollaries related to polynomials $A_{M-1}(z)$.

Corollary 1. Let $M, N \in \mathbb{N}_0 \setminus \{0\}$ and $z \neq 0$. Then we have

$$\frac{A_{M-1}(Nz)}{z^{M-1}} = \sum_{k=0}^{N-1}\sum_{j=0}^{M-1} \binom{M-1}{j} k^{M-1-j} \left(\frac{1-zk}{z}\right)^j A_j\left(\frac{z}{1-zk}\right).$$

Proof. We know that

$$\Phi(Nz, 1, M) = -(Nz)^{-M}\{\log(1-Nz) + A_{M-1}(Nz)\}.$$

By using Theorem 1,

$$N^{-M} \sum_{k=0}^{N-1}\sum_{j=0}^{M-1} \binom{M-1}{j} \frac{k^{M-1-j}}{1-zk} \Phi\left(\frac{z}{1-zk}, 1, j+1\right)$$
$$= -N^{-M} \sum_{k=0}^{N-1}\sum_{j=0}^{M-1} \binom{M-1}{j} \frac{k^{M-1-j}}{1-zk} \left(\frac{z}{1-zk}\right)^{-j-1} \left\{\log\left(1 - \frac{z}{1-zk}\right) + A_j\left(\frac{z}{1-zk}\right)\right\}$$
$$= -(Nz)^{-M}\{\log(1-Nz) + A_{M-1}(Nz)\}.$$

By using the binomial expansion, we get

$$\frac{A_{M-1}(Nz)}{z^{M-1}} = \frac{1}{z^{M-1}} \sum_{k=0}^{N-1} \log\left(1 - \frac{z}{1-zk}\right)$$
$$= \sum_{k=0}^{N-1} \left\{\sum_{j=0}^{M-1} \binom{M-1}{j} k^{M-1-j} \left(\frac{1-zk}{z}\right)^j\right\} \log\left(1 - \frac{z}{1-zk}\right).$$

Then, we arrive at the desired result. □

Corollary 2. Let $M \in \mathbb{N}_0 \setminus \{0\}$ and $az \neq 0$. Then we have

$$\frac{A_{M-1}(z)}{(az)^{M-1}} = \sum_{k=0}^{M-1} \binom{M-1}{k} (-1)^{M-k-1} \left(1 + \frac{1}{az}\right)^k \left\{A_k\left(\frac{az+z}{az+1}\right) - A_k\left(\frac{az}{az+1}\right)\right\}.$$

Proof. By using Theorem 2,

$$\Phi(z, 1, M) = -z^{-M}\{\log(1-z) + A_{M-1}(z)\}$$
$$= \frac{a^M}{1+az} \sum_{k=0}^{M-1} \binom{M-1}{k} (-1)^{M-k-1} \left\{-\left(\frac{a+1}{a}\right)^{k+1} \left(\frac{az+z}{az+1}\right)^{-k-1} \left(\log\left(1 - \frac{az+z}{az+1}\right) - A_k\left(\frac{az}{az+1}\right)\right)\right\}$$
$$+ \frac{a^M}{1+az} \sum_{k=0}^{M-1} \binom{M-1}{k} (-1)^{M-k-1} \left\{\left(\frac{az}{az+1}\right)^{-k-1} \left(\log\left(1 - \frac{az}{az+1}\right) - A_k\left(\frac{az}{az+1}\right)\right)\right\}$$
$$= \frac{a^M}{1+az} \sum_{k=0}^{M-1} \binom{M-1}{k} (-1)^{M-k-1} \left\{\begin{matrix} -\left(\frac{a+1}{a}\right)^{k+1} \left(\frac{az+1}{az+z}\right)^{k+1} \log\left(\frac{1-z}{az+1}\right) \\ + \left(\frac{az+1}{az}\right)^{k+1} \log\left(\frac{1}{az+1}\right) \end{matrix}\right\}$$
$$+ \frac{a^M}{1+az} \sum_{k=0}^{M-1} \binom{M-1}{k} (-1)^{M-k-1} \left\{\begin{matrix} -\left(\frac{a+1}{a}\right)^{k+1} \left(\frac{az+1}{az+z}\right)^{k+1} A_k\left(\frac{az+z}{az+1}\right) \\ + \left(\frac{az+1}{az}\right)^{k+1} A_k\left(\frac{az}{az+1}\right) \end{matrix}\right\}$$

After some elementary calculation, we get

$$-\left(\frac{a+1}{a}\right)^{k+1} \left(\frac{az+1}{az+z}\right)^{k+1} \log\left(\frac{1-z}{az+1}\right) + \left(\frac{az+1}{az}\right)^{k+1} \log\left(\frac{1}{az+1}\right)$$
$$= -\left(\frac{az+1}{az}\right)^{k+1} \log(1-z).$$

and

$$-\left(\frac{a+1}{a}\right)^{k+1}\left(\frac{az+1}{az+z}\right)^{k+1}A_k\left(\frac{az+z}{az+1}\right)+\left(\frac{az+1}{az}\right)^{k+1}A_k\left(\frac{az}{az+1}\right)$$

$$=-\left(\frac{az+1}{az}\right)^{k+1}\left\{A_k\left(\frac{az+z}{az+1}\right)-A_k\left(\frac{az}{az+1}\right)\right\}.$$

Therefore,

$$\Phi(z,1,M)$$

$$=-\frac{a^M}{1+az}\sum_{k=0}^{M-1}\binom{M-1}{k}(-1)^{M-k-1}\left(1+\frac{1}{az}\right)^{k+1}\log(1-z)$$

$$-\frac{a^M}{1+az}\sum_{k=0}^{M-1}\binom{M-1}{k}(-1)^{M-k-1}\left(1+\frac{1}{az}\right)^{k+1}\left\{A_k\left(\frac{az+z}{az+1}\right)-A_k\left(\frac{az}{az+1}\right)\right\}$$

By using the binomial expansion, we get

$$-\frac{a^M}{1+az}\sum_{k=0}^{M-1}\binom{M-1}{k}(-1)^{M-k-1}\left(1+\frac{1}{az}\right)^{k+1}\log(1-z)=-\frac{a^{M-1}}{z}\left(\frac{1}{az}\right)^{M-1}\log(1-z)$$

$$=-z^{-M}\log(1-z).$$

Then, we obtain the desired result by the following relation:

$$-z^{-M}A_{M-1}(z)$$

$$=-\frac{a^M}{1+az}\sum_{k=0}^{M-1}\binom{M-1}{k}(-1)^{M-k-1}\left(1+\frac{1}{az}\right)^{k+1}\left\{A_k\left(\frac{az+z}{az+1}\right)-A_k\left(\frac{az}{az+1}\right)\right\}.$$

□

Corollary 3. Let $M\in\mathbb{N}_0\setminus\{0\}$ and $z\neq 0$. Then we have

$$\frac{A_{M-1}(bz)-A_{M-1}(az)}{z^{M-1}}$$

$$=\sum_{j=0}^{M-1}\binom{M-1}{j}a^{M-1-j}\left(\frac{1-za}{z}\right)^{j}A_j\left(\frac{z(b-a)}{1-za}\right).$$

Proof. We know that

$$b^M\Phi(bz,1,M)-a^M\Phi(az,1,M)$$

$$=z^{-M}\{\log(1-az)-\log(1-bz)+A_{M-1}(az)-A_{M-1}(bz)\}.$$

By using Theorem 3,

$$\frac{(b-a)^M}{1-za}\sum_{j=0}^{M-1}\binom{M-1}{j}\left(\frac{a}{b-a}\right)^{M-1-j}\Phi\left(\frac{z(b-a)}{1-za},1,j+1\right)$$

$$=-\frac{(b-a)^M}{1-za}\sum_{j=0}^{M-1}\binom{M-1}{j}\left(\frac{a}{b-a}\right)^{M-1-j}\left\{\left(\frac{z(b-a)}{1-za}\right)^{-j-1}\left(\log\left(1-\frac{z(b-a)}{1-za}\right)+A_j\left(\frac{z(b-a)}{1-za}\right)\right)\right\}$$

$$=-\frac{1}{z}\sum_{j=0}^{M-1}\binom{M-1}{j}a^{M-1-j}\left(\frac{1-za}{z}\right)^{j}\log\left(\frac{1-bz}{1-az}\right)$$

$$-\frac{1}{z}\sum_{j=0}^{M-1}\binom{M-1}{j}a^{M-1-j}\left(\frac{1-za}{z}\right)^{j}A_j\left(\frac{z(b-a)}{1-za}\right).$$

By using the binomial expansion, we get

$$-z^{-M}\log\left(\frac{1-bz}{1-az}\right)$$

$$=-\frac{1}{z}\left\{\sum_{j=0}^{M-1}\binom{M-1}{j}a^{M-1-j}\left(\frac{1-za}{z}\right)^{j}\right\}\log\left(\frac{1-bz}{1-az}\right).$$

From the above relation, we arrive at the desired result. □

Corollary 4. Let $a, b, M \in \mathbb{N}_0 \setminus \{0\}$, $z \neq 0$ and $b > a > 0$. Then, we have

$$\sum_{k=a}^{b-1}\sum_{j=0}^{M-1}\binom{M-1}{j} k^{M-j-1}\left(\frac{1-zk}{z}\right)^j A_j\left(\frac{z}{1-zk}\right)$$

$$=\sum_{j=0}^{M-1}\binom{M-1}{j} a^{M-j-1}\left(\frac{1-za}{z}\right)^j A_j\left(\frac{z(b-a)}{1-za}\right).$$

Proof. By using Corollary 1, we have

$$\frac{A_{M-1}(bz)-A_{M-1}(az)}{z^{M-1}}$$

$$=\sum_{k=a}^{b-1}\sum_{j=0}^{M-1}\binom{M-1}{j} k^{M-1-j}\left(\frac{1-zk}{z}\right)^j A_j\left(\frac{z}{1-zk}\right).$$

If we use Corollary 3 into the above relation, we arrive at our final result. □

References

[1] A. Erdélyi, Higher Transcendental Functions, vol. 1, New York: McGrow-Hill, 1955.

[2] H. M. Srivastava, J. Choi, Series Associated with the Zeta and Related Functions, Kluwer Academic Publishers, Dordrecht, Boston and London, 2001.

[3] H. M. Srivastava, Some properties and results involving the zeta and associated functions, Functional Analysis, Approximation and Computation 7 (2) (2015) 89-133.

[4] E. C. Titchmarsh, The Theory of the Riemann-zeta function, Oxford University (Clarendon) Press, Oxford and London, 1951; Second Edition (Revised by D. R. Heath-Brown),1986.

[5] T. M. Apostol, Introduction to Analytic Number Theory, Springer-Verlag, New York, Heidelberg and Berlin, 1976.

[6] A. Bayad and Y. Simsek, Values of Twisted Barnes Zeta Functions at Negative Integers, Russian Journal of Mathematical Physics 20 (2) (2013) 129-137.

[7] H. M. Srivastava, T. Kim, and Y. Simsek, q-Bernoulli Numbers and Polynomials Associated with Multiple q-Zeta Functions and Basic L-series, Russian Journal of Mathematical Physics 12 (2) (2005) 241-268.

[8] Y. Simsek, On twisted q-Hurwitz zeta function and q-two-variable L-function, Applied Mathematics and Computation 187 (2007) 466-473.

[9] Y. Simsek, Twisted (h, q)-Bernoulli numbers and polynomials related to twisted (h, q)-zeta function and L-function, J. Math. Anal. Appl. 324 (2006) 790-804.

[10] L. Lewin, Polylogarithms and Associated Functions, New York: North-Holland, 1981.

[11] A. Erdélyi, W. Magnus, F. Oberhettinger, F. G. Tricomi, Higher Transcendental Functions, vol. 1, McGraw-Hill Book Company, New York, Toronto and London, 1953.

[12] J. Choi, Remark on the Hurwitz-Lerch zeta function, Fixed Point Theory and Applications 70 (2013) 2-10.

[13] A. A. Aygunes, Some identities for a special case of Hurwitz-Lerch zeta function, Numerical Analy. Appl. Math. AIP Conf. Proc.,accepted.

[14] M. G. Bin-Saad, Hypergeometric series associated with the Hurwitz-Lerch zeta function, Acta Math. Univ. Comenianae 2 (2009) 269-286.

[15] W. Kaplan, Advanced Calculus, Addison-Wesley, Reading, MA, 4th edition, 1993.